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Majorana neutrinos and their oscillations in QFT

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Tiivistelmä — Referat — Abstract

In this work we consider the method of unitarily inequivalent representations in the context of Majorana neutrinos and a simple seesaw model. In addition, the field theoretical framework of neutrino physics, namely that of QFT and the SM, is reviewed.

The oscillating neutrino states are expressed via suitable quantum operators acting on the physical vacuum of the theory, which provides further insight to the phenomenological flavor state ansatz made in the standard formulation of neutrino oscillations. We confirm that this method agrees with known results in the ultrarelativistic approximation while extending them to the non-relativistic region.

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Contents

1. Introduction

The neutrino holds a special place in the Standard Model (SM) of particle physics. It is the only fundamental fermion with zero mass. It has an electric charge of zero and it interacts via the weak force only. It always has a spin opposite to its momentum [1, 2, 3, 4]. And as we currently know, this picture is wrong.

The first ideas on neutrino oscillation, a phenomenon which implies a massive neutrino, were suggested in 1957-58 by B. Pontecorvo [5, 6], in a time when the common opinion was in favor of the massless neutrino [7]. This was followed by a decade when the electroweak (EW) unification progressed to what we now know as the SM of EW theory [8, 9], while the solar neutrino problem challenged the SM and backed up the oscillation theory [10, 11].

The first phenomenological model on neutrino mixing was proposed in 1969 [12], and by 1987 the standard neutrino oscillation theory had been elaborated [13]. In 1998, via atmospheric neutrino experiments [14], it was concluded that the oscillations exist in reality, undeniably cementing the neutrino as a particle whose behavior is inexplicable by the SM.

Nowadays, numerous extensions of the SM have been devised to account for neutrino mass [15, 16]. All of them have to fit into the constraint of Lorentz -invariance, which allows for two distinct types of neutrinos. Either the neutrino is a Dirac-type particle, its antineutrino counterpart a distinct particle from it. Or the neutrino is a Majorana-type particle and there is no distinct antineutrino, but lepton number conservation is broken. Which of these is the physical reality is an open question [17].

The questions on neutrino nature are not limited to whether it is Dirac or Majorana. The smallness of neutrino mass compared to other leptons is another: if the Higgs mechanism is the source of all masses, it is rather dissatisfactory from a theory perspective that some are orders of magnitude less than others. One popular answer is that indeed, the Higgs mechanism isn't enough by itself: the seesaw mechanism [18, 19, 20] predicts a Majorana-type neutrino, whose small mass is explained by one or more heavy particles in interaction with the Higgs and the neutrino.

In this work we extend the discussion of [21] to the case of the Majorana neutrino and a simple seesaw model. This is a well-known Hamiltonian diagonalization framework [22, 23], except in a new context. The oscillating neutrino states are expressed via suitable quantum operators acting on the physical vacuum of the theory, which provides further insight to the phenomenological flavor state ansatz made in the standard oscillation theory. We find results similar to [21] and confirm once more that this framework agrees with known results in the ultrarelativistic approximation while extending them to the non-relativistic region [24].

The work is organized as follows. In chapter 2 the quantum field theoretical formulation of spin 1/2 -particles is studied, followed by the electroweak theory in chapter 3. Chapter 4 goes through the standard phenomenological treatment of neutrino physics and chapter 5 presents the application of [21] to the Majorana neutrino.

2. Spin 1/2 particles in QFT

In order to understand neutrinos, one has to understand basic quantum field theory (QFT).

In this chapter the mathematical formalism of spin-1/2 -particles is reviewed, following mostly the texts [1, 2, 25, 26, 27, 28]. The Dirac Lagrangian describing such particles is introduced and its group theoretical foundations briefly reviewed. The quantized solutions to this equation are studied, concluding with the introduction of the Majorana field.

2.1 The Lorentz group

The Poincaré group, ISO(1,3), is the group of Minkowski spacetime isometries: linear transformations that leave a spacetime interval unchanged. It is the group of special relativity, containing rotations, boosts, and translations. Hence it is a group which has to be molded into any (special) relativistic theory of physics.

The Lorentz group, O(1,3), is a subgroup of the Poincaré group, it omits the translations.[†] It is also an example of a Lie group: a group with an infinite amount of elements, but a finite amount of generators. It is not simply connected nor compact, and for that reason we'll discuss the proper orthochronous, also called the restricted, Lorentz group $\mathrm{SO}^+(1,3)$, since it is the one that is most important for physics.[‡]

For any group, all group elements $g \in \mathcal{G}$ that are connected to the identity element can be expressed with the help of the generators as $g = \exp(i\theta_i^g \Lambda_i)$, where θ_i^g are real

[†]We remark that some authors, such as [26, Chapter 10.3], call the Lorentz group SO(1,3), when they actually talk about the restricted group. We stress that SO is a group whose elements have determinant +1.

[‡]The relation between these two groups is the isomorphy $SO^+(1,3) \cong O(1,3)/V_4$, where V_4 is the Klein 4-group $K_4 = \{I, P, T, PT\}$, in which I is identity, $P = \mathrm{diag}(1,-1,-1,-1)$ is spatial inversion, and $T = \mathrm{diag}(-1,1,1,1)$ the time-reversal. So understanding $\mathrm{SO}^+(1,3)$ is key to understanding the rest. The restricted group preserves both spatial and temporal orientation.

parameters and Λ_i are the generators. Any generator of a continuous symmetry of the action of a physical system* is associated with a conserved quantity as per Noether's theorem. In the case of ISO(1,3) the conserved quantities are the 4-momentum (translations) and the relativistic angular momentum (rotations and boosts), which highlights the physical importance of the group.

We can construct the rotations and the boosts of SO$^+$(1,3) using generators J_i and K_i respectively. The exact form of these generators depends on the representation, but there exist 3 of each type (follows from the dimension of the group) and satisfy the commutation relations

$$[J_i, J_j] = i\epsilon_{ijk}J_k, \tag{2.1}$$

$$[J_i, K_j] = i\epsilon_{ijk}K_k, \tag{2.2}$$

$$[K_i, K_j] = -i\epsilon_{ijk}J_k. \tag{2.3}$$

From these†, a crucial property can be shown: take

$$J_i^+ = \frac{1}{2}(J_i + iK_i), \tag{2.4}$$

$$J_i^- = \frac{1}{2}(J_i - iK_i). \tag{2.5}$$

With these, the commutation relations (2.1 - 2.3) become

$$[J_i^+, J_j^+] = i\epsilon_{ijk}J_k^+, \tag{2.6}$$

$$[J_i^-, J_j^-] = i\epsilon_{ijk}J_k^-, \tag{2.7}$$

$$[J_i^+, J_j^-] = 0, \tag{2.8}$$

which shows that the representations of $SU(2) \times SU(2)$ define the projective representations of $SO^+(1,3)$.‡

We note that expressing group elements by exponentiating these generators actually produces the group $SL(2,\mathbb{C})$, which is the universal cover of $SO^+(1,3)$. However, the projective representations of the Lorentz group agree with the representations of $SL(2,\mathbb{C})$, and the projective representations are the ones that agree with physics§, as long as we ensure the representation is also unitary.¶

Let us identify the physically relevant representation that corresponds to spin 1/2 -particles.

*In other words, a differentiable group action which leaves the Lagrangian invariant.

†These relations also define the Lorentz algebra $\mathfrak{so}(1,3;\mathbb{R})$.

‡In terms of algebras, we have $\mathfrak{so}(1,3;\mathbb{R}) \cong \mathfrak{sl}(2;\mathbb{C})$.

§The projective representations satisfy $r[g_1]r[g_2] = e^{i\phi(g_1,g_2)}r[g_1g_2]$. In a physical context we seek for representations that preserve the norm, and this allows for arbitrary phases.

¶For we also want matrix elements to be Poincaré invariant.

2.2 The spinor representation

From Wigner's theorem we know that a massive or massless spin 1/2 -particle has 2 degrees of freedom. Thus it can be described by a 2-component vector.* The unitary $SO^+(1,3)$ representation which acts upon these vectors shall built from 2x2 matrices, and the generators in this representation are related to the Pauli matrices (up to a scaling factor),

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.9}$$

and we denote them $\tau_i \equiv \frac{1}{2}\sigma_i$.

There exist two complex spin 1/2 representations, $(\frac{1}{2},0)_L$ and $(0,\frac{1}{2})_R$. These correspond to the combinations $J_i^- = \tau_i, J_i^+ = 0$ and $J_i^- = 0, J_i^+ = \tau_i$ respectively. We say that **spinors** are the elements of the vector space upon which these representations act. The elements of $(0,\frac{1}{2})_R$ are called **right-chiral Weyl spinors**, and the elements of $(\frac{1}{2},0)_L$ are called **left-chiral Weyl spinors**.

In terms of actual Lorentz transformations, these representations behave differently. Inverting the definitions (2.4, 2.5), we have

$$\left(\frac{1}{2},0\right)_L: \qquad J_i = \frac{1}{2}\sigma_i, \quad K_i = \frac{i}{2}\sigma_i \tag{2.10}$$

$$\left(0,\frac{1}{2}\right)_R: \qquad J_i = \frac{1}{2}\sigma_i, \quad K_i = -\frac{i}{2}\sigma_i. \tag{2.11}$$

Which establishes the Lorentz transformations for spinors. Let there be a field ψ_L which transforms under $(\frac{1}{2},0)_L$ and a field ψ_R which transforms under $(0,\frac{1}{2})_R$, and denote by $\theta,\beta \in \mathbb{R}$ the infinitesimal rotation and boost angles. Then

$$\begin{aligned} \psi_L &\to \exp\left(\frac{1}{2}[i\theta_j\sigma_j - \beta_j\sigma_j]\right)\psi_L \\ &= \left(\mathbb{1} + \frac{i}{2}\theta_j\sigma_j - \frac{1}{2}\beta_j\sigma_j\right)\psi_L, \end{aligned} \tag{2.12}$$

$$\begin{aligned} \psi_R &\to \exp\left(\frac{1}{2}[i\theta_j\sigma_j + \beta_j\sigma_j]\right)\psi_R \\ &= \left(\mathbb{1} + \frac{i}{2}\theta_j\sigma_j + \frac{1}{2}\beta_j\sigma_j\right)\psi_R. \end{aligned} \tag{2.13}$$

From this we can relate the two representations as complex conjugates (up to a transformation $i\sigma_2$). The Pauli matrices (2.9) satisfy $\sigma_2\sigma_i^*\sigma_2 = -\sigma_i$, and with this we find

*In the wide sense of the word vector, which will be specified below.

that the object $i\sigma_2\psi_L^*$ transforms as ψ_R:

$$\begin{aligned} i\sigma_2\psi_L^* \to & i\sigma_2\Big(\mathbb{1} - \frac{i}{2}\theta_j\sigma_j^* - \frac{1}{2}\beta_j\sigma_j^*\Big)\psi_L^* \\ &= i\sigma_2\Big(\mathbb{1} - \frac{i}{2}\theta_j\sigma_j^* - \frac{1}{2}\beta_j\sigma_j^*\Big)\sigma_2\sigma_2\psi_L^* \\ &= \Big(\mathbb{1} + \frac{i}{2}\theta_j\sigma_j + \frac{1}{2}\beta_j\sigma_j\Big)i\sigma_2\psi_L^*, \end{aligned} \tag{2.14}$$

establishing

$$\psi_L \to i\sigma_2\psi_L^* \tag{2.15}$$

as the operation which takes a left-chiral -field into a right-chiral field and vice versa. Interestingly this operation twice-applied does not return the original field:

$$i\sigma_2[i\sigma_2\psi_L^*]^* = -\psi_L. \tag{2.16}$$

This will be discussed again later.

From the relations (2.6-2.8) we further note that a particle transforming under $(\frac{1}{2}, 0)_L$ can not be the same as a particle transforming under $(0, \frac{1}{2})_R$, for an elementary particle is by definition a set of states which transforms into itself under irreducible (and unitary) representations of ISO(1,3) [1, Chapter 8.1], [25, Chapter 2.5].

2.3 Unitarity, Lagrangian and the Dirac equation

What we have established so far are the irreducible spinor representations. Unitarity is achieved with the method of induced representations [25, Chapter 2.5].

The only functions of 4-momentum p^μ that are left invariant by the restricted Lorentz group ($\mathrm{SO}^+(1,3)$) transformations are the square $p^2 = \eta_{\mu\nu}p^\mu p^\nu$, and for $p^2 \leq 0$ also the sign. We therefore categorize the action of the restricted Lorentz group in terms of different 4-momenta, of which the only physically reasonable combinations are $p^2 > 0, p^0 > 0; p^\mu = 0$; and $p^2 = 0, p^0 > 0$, corresponding to massive and massless particles, and the vacuum, respectively.

In this manner, the massive and massless particles end up transforming under their own *little groups*, also known as stabilizer subgroups, of $\mathrm{ISO}^+(1,3)$. Respectively, the groups are SO(3) and ISO(2), and these do have unitary representations. For a more extensive discussion, see [25], for example.

With the appropriate representation established, the next step we take is to construct a

Lorentz-invariant* Lagrangian density† that is Hermitian. It turns out that the simplest one is

$$\mathcal{L} = i\psi_R^\dagger \sigma_\mu \partial^\mu \psi_R + i\psi_L^\dagger \overline{\sigma}_\mu \partial^\mu \psi_L - m(\psi_R^\dagger \psi_L + \psi_L^\dagger \psi_R), \tag{2.17}$$

where we've introduced the shorthands

$$\sigma^\mu \equiv (\mathbb{1}_{2\text{x}2}, \sigma^i), \tag{2.18}$$

$$\overline{\sigma}^\mu \equiv (\mathbb{1}_{2\text{x}2}, -\sigma^i). \tag{2.19}$$

It is noteworthy that this Lagrangian couples the L- and R-chiral fields only through the mass-term. The fields are independent for $m = 0$.

We can further squeeze the notation by introducing the **Dirac field**

$$\psi \equiv \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}, \tag{2.20}$$

the **Dirac adjoint**

$$\overline{\psi} \equiv \psi^\dagger \gamma_0; \tag{2.21}$$

and the γ **-matrices**

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \overline{\sigma}^\mu & 0 \end{pmatrix} \tag{2.22}$$

which will be discussed further in a later section. With this notation, the Lagrangian (2.17) becomes

$$\mathcal{L} = \overline{\psi}(i\gamma^\mu \partial_\mu - m\mathbb{1}_{4\text{x}4})\psi, \tag{2.23}$$

which is the familiar form of the Dirac Lagrangian. Taking the functional derivative and setting it to zero,

$$\frac{\partial \mathcal{L}}{\partial \psi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \right) = 0, \tag{2.24}$$

also known as the Euler-Lagrange equations, yields the equations of motion

$$(i\gamma^\mu \partial_\mu - m\mathbb{1}_{4\text{x}4})\psi = 0, \tag{2.25}$$

which are also known as the **Dirac equation**. These equations in terms of the fields ψ_L, ψ_R, which could also be retrieved from the Lagrangian (2.17), read as

$$i\sigma^\mu \partial_\mu \psi_R - m\psi_L = 0, \tag{2.26}$$

$$i\overline{\sigma}^\mu \partial_\mu \psi_L - m\psi_R = 0. \tag{2.27}$$

*To ensure the resulting equations of motion (the Euler-Lagrange -equations) are also Lorentz-invariant.

†The Lagrangian density $\mathcal{L}$ and the Lagrangian $L = \int d^3x \mathcal{L}(\mathbf{x}, t)$ are different objects, yet it is common to speak of both as the Lagrangian.

2.4 Dirac representation and the chiral basis

In section 2.3, we introduced the γ -matrices

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1}_{2x2} \\ \mathbb{1}_{2x2} & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \tag{2.28}$$

and promised a proper introduction. In defining the Dirac field (2.20) we introduced, without mentioning it, the direct sum representation $(\frac{1}{2}, 0)_L \bigoplus (0, \frac{1}{2})_R$ of the group $SU(2) \times SU(2)$. This is a 4-dimensional representation of the Lorentz algebra, which is defined by [2, Chapter 3.2]

$$S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu], \tag{2.29}$$

in which the matrices γ satisfy the anticommutation relations

$$\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu} \times \mathbb{1}_{4x4}. \tag{2.30}$$

It turns out that the γ -matrices introduced in (2.22) satisfy the relations (2.30). There are other bases for the 4-dimensional Dirac representation, such as the Dirac basis and Majorana basis, but in this thesis we'll stick with the expression (2.22) which is also known by the names of chiral basis or Weyl basis. Sometimes instead of bases, people call them representations, as in "Weyl representation", but this term overlaps with the group theoretical representation and shall thus be avoided.

In the chiral basis, the Lorentz boost and rotation generators are

$$S^{0i} = \frac{i}{4}[\gamma^0, \gamma^i] = -\frac{i}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}, \tag{2.31}$$

$$S^{ij} = \frac{i}{4}[\gamma^i, \gamma^j] = \frac{1}{2}\epsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}, \tag{2.32}$$

which manifests the difference in how L and R -chiral fields transform.

We can now make a comment to the observation made in eqn. (2.16), that the operation $\psi_L \to [i\sigma_2\psi_L^*]_R$ brings up an extra minus sign if applied twice. Using the Dirac fields and the matrix γ_2, we may construct an operation that achieves the same L-to-R -chiral transformation, but without the additional signs.

$$\begin{aligned} \psi \to i\gamma_2\psi^* &= \begin{pmatrix} i\sigma_2\psi_R^* \\ -i\sigma_2\psi_L^* \end{pmatrix} \\ &\to \begin{pmatrix} -i\sigma_2[i\sigma_2\psi_L^*]^* \\ -i\sigma_2[i\sigma_2\psi_R^*]^* \end{pmatrix} \\ &= \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}. \end{aligned} \tag{2.33}$$

In the Weyl basis, we also have the convenient expression for Weyl spinors, despite the slight abuse of notation:

$$\psi_L = \begin{pmatrix} \psi_1 \\ \psi_2 \\ 0 \\ 0 \end{pmatrix} \qquad \psi_R = \begin{pmatrix} 0 \\ 0 \\ \psi_3 \\ \psi_4 \end{pmatrix}. \tag{2.34}$$

Often it is clear from context whether ψ_L or ψ_R refer to the 2-component spinors introduced in section 2.2, or the 4-component spinors given by (2.34).

We can also define the fifth γ -matrix,*

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3, \tag{2.35}$$

which satisfies $(\gamma^5)^\dagger = \gamma^5, \{\gamma^5, \gamma^\mu\} = 0$, and in the chiral basis

$$\gamma^5 = \begin{pmatrix} -\mathbb{1}_{2\text{x}2} & 0 \\ 0 & \mathbb{1}_{2\text{x}2} \end{pmatrix} \tag{2.36}$$

and $(\gamma^5)^2 = \mathbb{1}_{4\text{x}4}$. This matrix can be used to construct the chiral projection operators

$$P_L \equiv \frac{\mathbb{1}_{4\text{x}4} - \gamma^5}{2}, \tag{2.37}$$

$$P_R \equiv \frac{\mathbb{1}_{4\text{x}4} + \gamma^5}{2}, \tag{2.38}$$

which indeed satisfy the properties of a projection: $P_L^2 = P_L, P_R^2 = P_R$, as well as $P_L\psi_L = \psi_L, P_L\psi_R = 0, P_R\psi_L = 0$, and $P_R\psi_R = \psi_R$, where fields ψ_L, ψ_R are as in (2.34).

2.5 Solutions of the Dirac equation

The Dirac equation, (2.25), has been solved in virtually every elementary QFT textbook and beyond. Now that we have a basis, we can proceed to list the main insights and results.

The analysis usually begins by noting that any Dirac field satisfying the equation (2.25) also satisfies the Klein-Gordon -equation:

$$(i\gamma^\nu\partial_\nu + m\mathbb{1}_{4\text{x}4})(i\gamma^\mu\partial_\mu - m\mathbb{1}_{4\text{x}4})\psi = (\partial^2 + m^2)\psi = 0, \tag{2.39}$$

*The number 5 is a relic from a time when the γ's were labeled from 1 to 4.

where $\partial^2 \equiv \partial_\mu \partial^\mu$. Eq. (2.39) then implies that the solutions are linear combinations of plane waves:

$$\psi \equiv \psi(x) = \int \frac{d^3p}{(2\pi)^3} u(\mathbf{p}) e^{-ipx}, \tag{2.40}$$

where $px \equiv p \cdot x \equiv p_\mu x^\mu$, $p^2 = m^2$, $p_0 = \sqrt{\mathbf{p}^2 + m^2} > 0$, $\mathbf{p}$ is the 3-momentum, and $u(\mathbf{p})$ is a 4-component **Dirac spinor**. Similarly there exist antiparticle solutions that correspond to $p_0 < 0$:

$$\chi(x) = \int \frac{d^3p}{(2\pi)^3} v(\mathbf{p}) e^{ipx}, \tag{2.41}$$

where now $v(\mathbf{p})$ is another 4-component (antiparticle) Dirac spinor.

The plane-wave solutions are then plugged into (2.25) to retrieve an equation for the Dirac spinors. In the chiral basis

$$\begin{pmatrix} -m & \sigma^\mu p_\mu \\ \overline{\sigma}^\mu p_\mu & -m \end{pmatrix} u(\mathbf{p}) = 0, \qquad \begin{pmatrix} m & \sigma^\mu p_\mu \\ \overline{\sigma}^\mu p_\mu & m \end{pmatrix} v(\mathbf{p}) = 0. \tag{2.42}$$

From here, one common textbook-approach is to consider the rest-frame, in which $p_\mu = (m, 0, 0, 0)$. This leads to

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} u(\mathbf{0}) = 0, \qquad \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} v(\mathbf{0}) = 0, \tag{2.43}$$

from which

$$u(\mathbf{0}) = \begin{pmatrix} \xi \\ \xi \end{pmatrix}, \qquad v(\mathbf{0}) = \begin{pmatrix} \eta \\ -\eta \end{pmatrix}, \tag{2.44}$$

which gives the insight that ξ and η are arbitrary and constant 2-component spinors. These are then boosted to a general frame, in which

$$u(\mathbf{p}) = \begin{pmatrix} \sqrt{p \cdot \sigma}\, \xi \\ \sqrt{p \cdot \overline{\sigma}}\, \xi \end{pmatrix}, \qquad v(\mathbf{p}) = \begin{pmatrix} \sqrt{p \cdot \sigma}\, \eta \\ -\sqrt{p \cdot \overline{\sigma}}\, \eta \end{pmatrix}. \tag{2.45}$$

Here the object $\sqrt{p \cdot \sigma}$ is defined by changing to a basis where $p \cdot \sigma$ is diagonal, taking the square root of the eigenvalues, and then changing back. [1, Chapter 11.2]

There exist 4 independent solutions for the Dirac spinors. These correspond to particles and antiparticles with two possible states each, in accord with the amount of degrees of freedom that correspond to spin 1/2 particles.

2.5.1 Helicity eigenstates

There exists a particularly useful choice of the 2-spinors introduced in (2.44): the eigenspinors of the helicity operator [29].

The helicity operator Σ_s is defined as

$$\Sigma_s \equiv \frac{\sigma}{2} \cdot \hat{p} = \begin{pmatrix} \cos(\theta) & \sin(\theta)\, e^{-i\phi} \\ \sin(\theta)\, e^{i\phi} & -\cos(\theta) \end{pmatrix}, \tag{2.46}$$

in which the dot product has been expressed in spherical coordinates, $\sigma/2$ is the spin operator, and $\hat{p}$ is a momentum 3-vector of unit length. This operator has the following eigenspinors:

$$\chi^{\uparrow} = \chi^{R} = \begin{pmatrix} \cos\frac{\theta}{2}\, e^{-i\frac{\phi}{2}} \\ \sin\frac{\theta}{2}\, e^{i\frac{\phi}{2}} \end{pmatrix}, \tag{2.47}$$

$$\chi^{\downarrow} = \chi^{L} = \begin{pmatrix} \sin\frac{\theta}{2}\, e^{-i\frac{\phi}{2}} \\ -\cos\frac{\theta}{2}\, e^{i\frac{\phi}{2}} \end{pmatrix}, \tag{2.48}$$

with eigenvalues +1/2 and -1/2, corresponding to up and down, or right and left, respectively.

Consider then eqns. (2.45) as an example. Choosing $\xi = \chi^{\uparrow}$ in the particle-spinors yields two independent solutions:

$$u^{\uparrow}(\mathbf{p}) = \begin{pmatrix} \sqrt{E - \sigma_i p^i}\, \chi^{\uparrow} \\ \sqrt{E + \sigma_i p^i}\, \chi^{\uparrow} \end{pmatrix} = \begin{pmatrix} \sqrt{E - |\mathbf{p}|}\, \chi^{\uparrow} \\ \sqrt{E + |\mathbf{p}|}\, \chi^{\uparrow} \end{pmatrix}, \tag{2.49}$$

$$u^{\downarrow}(\mathbf{p}) = \begin{pmatrix} \sqrt{E - \sigma_i p^i}\, \chi^{\downarrow} \\ \sqrt{E + \sigma_i p^i}\, \chi^{\downarrow} \end{pmatrix} = \begin{pmatrix} \sqrt{E + |\mathbf{p}|}\, \chi^{\downarrow} \\ \sqrt{E - |\mathbf{p}|}\, \chi^{\downarrow} \end{pmatrix}. \tag{2.50}$$

In this thesis, we will use a set of Dirac spinors that are listed in Appendix A.

A remark on spin, helicity and chirality

These three concepts are often mixed, but they are theoretically distinct:

Spin: Denotes the representation of the Lorentz group under which the given field transforms.
Chirality: A label given to the two irreducible spinor representations of the Lorentz group.*
Helicity: The projection of a particle's spin along the particle's momentum.

*Note that this label could be sensibly defined to any group which decomposes into representations of form (A,0)⊕(0,B).

2.5.2 The quantized Dirac field

The canonical quantization procedure involves the Hamiltonian density

$$\begin{aligned}\mathcal{H} &\equiv \frac{\partial \mathcal{L}}{\partial[\partial_0\psi]}[\partial_0\psi] - \mathcal{L} \\ &= \overline{\psi}(x)(-i\gamma^i\partial_i + m)\psi(x).\end{aligned} \tag{2.51}$$

We note that the canonical conjugate momentum to ψ is $i\psi^\dagger$, and label the differential operator $-i\gamma^i\partial_i + m = h_D$. Our goal is to express this Hamiltonian with appropriate quantum operators and to diagonalize it.

The spin-statistics theorem tells us that spin 1/2 -particles obey fermionic statistics [2]. Hence in our attempt to model those particles we impose the equal-time fermionic anticommutation relations

$$\{\psi_\alpha(\mathbf{x}), \psi^\dagger_\beta(\mathbf{y})\} \equiv \psi_\alpha(\mathbf{x})\psi^\dagger_\beta(\mathbf{y}) + \psi^\dagger_\beta(\mathbf{y})\psi_\alpha(\mathbf{x}) = \delta^3(\mathbf{x}-\mathbf{y})\delta_{\alpha\beta}, \tag{2.52}$$

$$\text{Other commutators zero,} \tag{2.53}$$

where α, β are spinor indices and $\delta_{\alpha\beta}$ is the Kronecker delta.

These commutation relations are realized on a quantum level by expanding $\psi(x)$ in the basis of eigenfunctions of h_D and including quantum operators as expansion coefficients. In section 2.5.1 we saw that there are two possible helicity eigenstates for a Dirac spinor. Thus, in the Schrödinger picture*, the appropriate expansion over all states is

$$\psi(\mathbf{x}) \equiv \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_\lambda \Big(a^\lambda_{\mathbf{p}} u^\lambda(\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}} + b^{\lambda\dagger}_{\mathbf{p}} v^\lambda(\mathbf{p})e^{-i\mathbf{p}\cdot\mathbf{x}}\Big), \tag{2.54}$$

where the sum is over helicity eigenstates, $a^\lambda_{\mathbf{p}}$ corresponds to particles and $b^\lambda_{\mathbf{p}}$ to antiparticles.

Applying (2.54) into (2.52, 2.53) yields the correct equal-time anticommutation relations for fermionic creation and annihilation operators:

$$\{a^\lambda_{\mathbf{p}}, a^{\lambda'\dagger}_{\mathbf{q}}\} = (2\pi)^3\delta^3(\mathbf{p}-\mathbf{q})\delta^{\lambda\lambda'}, \tag{2.55}$$

$$\{b^\lambda_{\mathbf{p}}, b^{\lambda'\dagger}_{\mathbf{q}}\} = (2\pi)^3\delta^3(\mathbf{p}-\mathbf{q})\delta^{\lambda\lambda'}, \tag{2.56}$$

$$\text{Other commutators zero.} \tag{2.57}$$

Note that these relations can also be realized without the constant $(2\pi)^3$, as is the case in [30, Chapter 4.2], which is the convention we use in Appendix B.

*In which ψ does not depend on time.

Using the Hamiltonian $H = \int d^3x\mathcal{H}$, the Dirac delta (B.7), the spinors listed in Appendix A, and normal ordering, we find*

$$H = \int \frac{d^3p}{(2\pi)^3} \sum_\lambda E_p \Big(a_{\mathbf{p}}^{\lambda\dagger} a_{\mathbf{p}}^{\lambda} + b_{\mathbf{p}}^{\lambda\dagger} b_{\mathbf{p}}^{\lambda} \Big), \tag{2.58}$$

which satisfies the property of being diagonal in terms of the operators:

$$H \propto \begin{pmatrix} a_{\mathbf{p}}^{\lambda\dagger} & b_{\mathbf{p}}^{\lambda\dagger} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_{\mathbf{p}}^{\lambda} \\ b_{\mathbf{p}}^{\lambda} \end{pmatrix}. \tag{2.59}$$

To ensure the system described has a minimum energy state, we postulate the existence of a vacuum state $|0\rangle$ that satisfies, for every $\mathbf{p}$ and λ,

$$a_{\mathbf{p}}^{\lambda}|0\rangle = b_{\mathbf{p}}^{\lambda}|0\rangle = 0. \tag{2.60}$$

We also define the one-particle -state

$$|a(\mathbf{p},\lambda)\rangle \sim a_{\mathbf{p}}^{\lambda\dagger}|0\rangle, \tag{2.61}$$

which, more explicitly, satisfy [2, Chapter 3.5]

$$\langle a(\mathbf{p},\lambda)|a(\mathbf{q},\lambda')\rangle = E_{\mathbf{p}}(2\pi)^3\delta^3(\mathbf{p}-\mathbf{q})\delta^{\lambda\lambda'}, \tag{2.62}$$

which, in other words, implies the action of $a_{\mathbf{p}}^{\lambda}$ is to reduce a one-particle state to the vacuum:

$$\begin{aligned} \langle a(\mathbf{q},\lambda')|a(\mathbf{p},\lambda)\rangle &= \langle 0|a_{\mathbf{q}}^{\lambda'}|a(\mathbf{p},\lambda)\rangle \\ &= 0, \text{unless } a_{\mathbf{q}}^{\lambda'}|a(\mathbf{p},\lambda)\rangle \sim |0\rangle. \end{aligned} \tag{2.63}$$

With these definitions and properties, we formalized the interpretation that $a_{\mathbf{p}}^{\lambda\dagger}$ creates a particle of momentum $\mathbf{p}$ and helicity λ, and $a_{\mathbf{p}}^{\lambda}$ annihilates a particle of momentum $\mathbf{p}$ and helicity λ. Operators b do the same with antiparticles.

We could've approached these calculations with the anticommutator as well as the definition of the vacuum state:

$$a_{\mathbf{q}}^{\lambda'}|a(\mathbf{p},\lambda)\rangle \sim a_{\mathbf{q}}^{\lambda'}a_{\mathbf{p}}^{\lambda\dagger}|0\rangle = \Big([2\pi]^3\delta^3(\mathbf{p}-\mathbf{q})\delta^{\lambda\lambda'} - a_{\mathbf{p}}^{\lambda\dagger}a_{\mathbf{q}}^{\lambda'}\Big)|0\rangle = (2\pi)^3\delta^3(\mathbf{p}-\mathbf{q})\delta^{\lambda\lambda'}|0\rangle. \tag{2.64}$$

We remark that the exclusion principle is manifest in the operators as it should:

$$a_{\mathbf{p}}^{\lambda\dagger}|a(\mathbf{p},\lambda)\rangle \sim a_{\mathbf{p}}^{\lambda\dagger}a_{\mathbf{p}}^{\lambda\dagger}|0\rangle = \frac{1}{2}\{a_{\mathbf{p}}^{\lambda\dagger}, a_{\mathbf{p}}^{\lambda\dagger}\}|0\rangle = 0. \tag{2.65}$$

*See Appendix B for a calculation which contains all the abovementioned elements more explicitly.

As a last step, we restore the time-dependence of the fields, by going to the Heisenberg picture, with

$$e^{iHt}a_{\mathbf{p}}^{\lambda}e^{-iHt} = a_{\mathbf{p}}^{\lambda}e^{-iE_pt}, \tag{2.66}$$
$$e^{iHt}b_{\mathbf{p}}^{\lambda\dagger}e^{-iHt} = b_{\mathbf{p}}^{\lambda\dagger}e^{iE_pt}, \tag{2.67}$$

which leads to

$$\psi(\mathbf{x},t) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_{\lambda}\Big(a_{\mathbf{p}}^{\lambda}u^{\lambda}(\mathbf{p})e^{-ipx} + b_{\mathbf{p}}^{\lambda\dagger}v^{\lambda}(\mathbf{p})e^{ipx}\Big), \tag{2.68}$$
$$\overline{\psi}(\mathbf{x},t) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_{\lambda}\Big(b_{\mathbf{p}}^{\lambda}\overline{v}^{\lambda}(\mathbf{p})e^{-ipx} + a_{\mathbf{p}}^{\lambda\dagger}\overline{u}^{\lambda}(\mathbf{p})e^{ipx}\Big). \tag{2.69}$$

2.6 Discrete transformations

Now that we've established the quantized Dirac field, we can study its discrete transformations. Note that the interpretations of this section, of given forms of P and C as parity and charge conjugation, are sensible only when applied to 4-component Dirac spinors.

2.6.1 Charge conjugation

The charge conjugation is an operation that takes particles to antiparticles while retaining the spin orientation. [2, Chapter 3.6]

This can be achieved by a quantum operator, $\hat{\mathcal{C}}$, acting upon creation and annihilation operators, or by a classical operation $\psi \to C\overline{\psi}^T$, where C is a 4x4 matrix acting upon the Dirac spinors u and v.

The quantum operator is defined by the relations

$$\hat{\mathcal{C}}a_{\mathbf{p}}^{\lambda}\hat{\mathcal{C}}^{-1} = b_{\mathbf{p}}^{\lambda}e^{i\phi(\mathbf{p},\lambda)}, \tag{2.70}$$
$$\hat{\mathcal{C}}b_{\mathbf{p}}^{\lambda\dagger}\hat{\mathcal{C}}^{-1} = a_{\mathbf{p}}^{\lambda\dagger}e^{i\phi(\mathbf{p},\lambda)}, \tag{2.71}$$

where $\phi(\mathbf{p},\lambda)$ denotes an arbitrary phase, which we choose to be zero. The explicit form of the quantum operator is shown in [27, chapter 15.12] for example.

The matrix C is defined by

$$C\gamma_{\mu}C^{-1} = -\gamma_{\mu}^{T}, \tag{2.72}$$

and in the chiral basis we find $C = i\gamma^2\gamma^0$. Hence the classical operation is

$$\psi \to C\overline{\psi}^T = -i\gamma^2\gamma^0\overline{\psi}^T = -i\gamma^2\psi^*, \tag{2.73}$$

and the matrix C is found to satisfy

$$C = -C^{-1} = -C^{T} = -C^{\dagger}. \tag{2.74}$$

Applying the expansions (2.68, 2.69) and the relation (A.7) one can verify that the classical and quantum operations agree:

$$\hat{\mathcal{C}}\psi(\mathbf{x},t)\hat{\mathcal{C}}^{-1} = C\overline{\psi}^{T}(\mathbf{x},t). \tag{2.75}$$

The classical operation is, up to a sign, the same L- to R-chiral transformation of eqns. (2.33), which clarifies the interpretation of that relation.

Oftentimes the notation ψ^{c} is used when denoting a charge conjugation of ψ.

2.6.2 Parity

The parity operation, P, is equated with a spatial inversion. We define its action upon the coordinate system as

$$P : (\mathbf{x},t) \to (-\mathbf{x},t). \tag{2.76}$$

The spatial inversion is a part of the full Lorentz group, but not the proper orthochronous -group, for it is a discrete transformation. On physical grounds, we can impose properties to this operation when applying it to fields. For example, P is a known symmetry of quantum electrodynamics (QED), and we shall use this to construct the operator.

Before that, we consider the case of free scalar fields to illustrate a known ambiguity in this definition, following [1, 25]. This discussion also shows that the parity operation of a given theory is not necessarily the trivial spatial inversion, which satisfies $P^2 = 1$.

The free scalar Lagrangian is $\mathcal{L} = -\frac{1}{2}\phi^{*}\Box\phi - \frac{1}{2}m^2|\phi|^2$. This should exhibit invariance upon parity, which leads to the definition

$$P : \phi(\mathbf{x},t) \to \eta_P\phi(-\mathbf{x},t), \tag{2.77}$$

where η_P is a phase, called the intrinsic parity of ϕ. However, the free Lagrangian is also invariant under any internal transformation* which takes $\phi(\mathbf{x},t) \to e^{i\alpha}\phi(\mathbf{x},t)$. Such invariances in an actual physical theory would correspond, for example, to the conservation of lepton number in the case of a Dirac Lagrangian.

Therefore, if we use the argument of "symmetry under parity" to define how parity acts upon fields, we could equivalently define $P' : \phi(\mathbf{x},t) \to \eta_P e^{i\alpha}\phi(-\mathbf{x},t) \equiv \eta_{(P')}\phi(-\mathbf{x},t)$.

*An internal transformation affects only the fields, not the spacetime coordinates.

Redefining the parity operator in this manner can lead to cases where $P'^2 \neq \pm 1$, depending on the theory at hand [25].

In any case, we have a choice to make: choosing the intrinsic parity of one type of a field fixes the intrinsic parities of other fields as well. If there are more fields and more internal transformations that keep $\mathcal{L}$ invariant, we can choose the intrinsic parities of as many fields as we have transformations.

With these remarks out of the way, we recite the way parity works for Dirac spinors, following [2].

Parity flips the momentum, but keeps spin invariant. Acting upon quantum operators, we have

$$\mathcal{P}a^\lambda(\mathbf{p})\mathcal{P}^{-1} = \eta_a a^{-\lambda}(-\mathbf{p}), \tag{2.78}$$

$$\mathcal{P}b^\lambda(\mathbf{p})\mathcal{P}^{-1} = \eta_b b^{-\lambda}(-\mathbf{p}), \tag{2.79}$$

where η_a, η_b are phases that are shown to satisfy $\eta_a^2, \eta_b^2 \in \{-1, +1\}$, as well as $\eta_a = -\eta_b^*$. We remark that λ refers to helicity, not spin, in our notation. Again, for the explicit form of the quantum operator, the reader is referred to [27, Chapter 15.11].

The classical operation is found to be

$$\psi(\mathbf{x}, t) \to \eta_a \gamma^0 \psi(-\mathbf{x}, t), \tag{2.80}$$

where, in the context of Dirac fields, we may choose $\eta_a = +1$. In the case of Majorana fields, which are introduced in the next section, the intrinsic parity is found to be $\pm i$.

Again both realizations agree with eachother:

$$\mathcal{P}\psi(\mathbf{x}, t)\mathcal{P}^{-1} = \gamma^0 \psi(-\mathbf{x}, t). \tag{2.81}$$

2.7 Majorana fields

Now that we've discussed the Dirac field and its charge conjugation, we are ready to introduce the Majorana field into this mathematical framework.

Recall that in constructing the Dirac Lagrangian, our guiding principle was Lorentz-invariance. The mass-terms demanded a combination of a field that transforms under $(\frac{1}{2}, 0)$, and a field that transforms under $(0, \frac{1}{2})$. It turns out that we may define a Lorentz-invariant Lagrangian using only L-chiral fields, by constructing an R-chiral field from them using (2.15). We call the following Lagrangian a Majorana Lagrangian:

$$\mathcal{L}_{\text{Majorana}} = i\Big(\psi_L^\dagger \overline{\sigma}_\mu \partial^\mu \psi_L + \frac{m}{2}[\psi_L^\dagger \sigma_2 \psi_L^* - \psi_L^T \sigma_2 \psi_L]\Big). \tag{2.82}$$

This leads to the equations of motion [28, Chapter 6.2]

$$\overline{\sigma}_\mu \partial^\mu \psi_L + m\sigma_2\psi_L^* = 0, \tag{2.83}$$

which are known as the Majorana equation.

We can express the above with the Dirac 4-spinor formalism as well. Building a Dirac field with the above prescription, we have

$$\psi^M = \begin{pmatrix} \psi_L \\ i\sigma_2\psi_L^* \end{pmatrix}, \tag{2.84}$$

which, when inserted into the Dirac Lagrangian (2.23), results in (2.82). The superscript denotes the fact that this is a Majorana field which is embedded into a Dirac field.

From (2.84) we readily see that this is a field that is its own charge conjugate:

$$C\overline{[\psi^{\mathrm{M}}]}^T = \psi^{\mathrm{M}}. \tag{2.85}$$

Oftentimes, this relation is stated to be the definition of a Majorana field, or equivalently: Majorana particles are their own antiparticles. Therefore, they have to be neutral particles.

Considering the restriction (2.85) with the quantized Dirac field expanded in momentum space, (2.68), we are led to the momentum expansion of the Majorana field*:

$$\psi^{\mathrm{M}}(\mathbf{x}, t) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_\lambda \Big(a_{\mathbf{p}}^\lambda u^\lambda(\mathbf{p}) e^{-ipx} + a_{\mathbf{p}}^{\lambda\dagger} v^\lambda(\mathbf{p}) e^{ipx} \Big). \tag{2.86}$$

*Equivalently, we could derive this with the canonical quantization procedure, starting from the Lagrangian (2.82).

3. The SM of EW interactions

The standard model of electroweak interactions, known also as the Glashow-Weinberg-Salam -theory, developed in the late 1960's by the aforementioned and proved renormalizable in early 1970's by 't Hooft and Veltman, unifies weak interactions and quantum electrodynamics (QED). It is based upon the principle of gauge- and Lorentz invariance and employs a specific symmetry breaking scheme known as the Brout-Englert-Higgs mechanism.

In this chapter, we review this theory, following mostly the books [1, 2, 3, 4, 25].

3.1 Gauge field theories

By gauge field theories, we mean theories that feature *gauge invariant* Lagrangians, i.e. Lagrangians that are invariant under certain *local* transformations that are not Lorentz-transformations.

Often the case is, in gauge field theories, that the mathematical description introduces an excess of parameters when compared to the physical degrees of freedom (dof). We'll see this first in the case of electrodynamics, where the photon (2 dof, the polarizations) is described with a 4-vector field (4 dof) to ensure Lorentz invariance. A process called *gauge fixing* is used to eliminate the additional, nonphysical dof.

We'll introduce QED with some intuition from classical physics. However, the principle of gauge invariance produces the exact same results. We'll elevate gauge invariance to a first principle, much alike Lorentz-invariance, to introduce the Yang-Mills theories. We then proceed into the theory closest to the neutrino, the electroweak theory.

3.1.1 QED and gauge invariance

In chapter 2, we established the free Dirac Lagrangian for massive spin 1/2 particles, eqn. (2.23). This constitutes one part of QED. What remains is to introduce the quantized massless spin 1 particles, photons, to the theory. We will do so following [25, Chapter 5.9] and [1, Chapter 8.2.3.].

Recall from Wigner's theorem that a massless spin 1 particle has 2 dof, and thus the least complicated field we can use to describe such is a 4-vector field. However, no quantized 4-vector field can be constructed from the creation and annihilation operators of a massless particle of helicity ± 1.

We have to somehow match the degrees of freedom of a 4-vector field to that of a massless spin 1 particle. This is accomplished by introducing a polarization vector $\epsilon^\mu(\mathbf{p}, \lambda)$. The ansatz for a photon field mode expansion is

$$A^\mu(x) \equiv \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega}} \sum_\lambda \left[\epsilon^\mu(\mathbf{p}, \lambda) a_{\mathbf{p}}^\lambda e^{ipx} + \epsilon^{\mu *}(\mathbf{p}, \lambda) a_{\mathbf{p}}^{-\lambda \dagger} e^{-ipx} \right], \tag{3.1}$$

where the sum is over the two possible helicities and it satisfies the massless Klein-Gordon equation $\partial^\mu \partial_\mu A^\nu(x) = 0$.

From classical electrodynamics, we know light is transverse-polarized, so we realign our coordinates so that the photon moves in the z-direction, and postulate

$$\epsilon^0(\mathbf{p}, \pm 1) = 0, \tag{3.2}$$

$$\mathbf{p} \cdot \epsilon(\mathbf{p}, \pm 1) = 0. \tag{3.3}$$

It follows that

$$A^0(x) = 0, \tag{3.4}$$

$$\nabla \cdot \mathbf{A}(x) = 0. \tag{3.5}$$

In particular, $A^0(x) = 0$, in all frames, shows that $A^\mu(x)$ is not a 4-vector. Despite this, we can still use it to construct an antisymmetric tensor field

$$F^{\mu\nu}(x) \equiv \partial^\mu A^\nu(x) - \partial^\nu A^\mu(x), \tag{3.6}$$

which can be shown to satisfy the vacuum Maxwell equations

$$\partial_\mu F^{\mu\nu} = 0, \tag{3.7}$$

$$\varepsilon^{\rho\sigma\mu\nu} \partial_\sigma F_{\mu\nu} = 0, \tag{3.8}$$

and are also the simplest Lorentz-covariant fields that correspond to a massless particle of helicity ± 1.* Hence, the simplest Lorentz-invariant scalar corresponding to the

*The discussion at the end of [25, Chapter 5.9] is particularly enlightening.

photon is $F_{\mu\nu}F^{\mu\nu}$, and we use it to build the quantized counterpart of the Lagrangian of electromagnetism,

$$\mathcal{L}_{\text{EM}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{3.9}$$

Thus, QED without interactions is

$$\mathcal{L} = \mathcal{L}_{\text{EM}} + \mathcal{L}_{\text{Dirac}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}(i\gamma^\mu\partial_\mu - m\mathbb{1}_{4\text{x}4})\psi. \tag{3.10}$$

From this, we remark: $\mathcal{L}_{\text{EM}}$ is invariant under a transformation

$$A_\mu(x) \to A_\mu(x) + \partial_\mu\alpha(x), \tag{3.11}$$

for any function $\alpha(x)$. On the other hand $\mathcal{L}_{\text{Dirac}}$ is invariant under a *global* U(1) transformation $\psi \to e^{i\theta\beta}\psi$, but yields an additional term under a *local* U(1) transformation $\psi \to e^{ie\beta(x)}\psi$:

$$\mathcal{L}_{\text{Dirac}} \to \overline{\psi}(i\gamma^\mu\partial_\mu - m\mathbb{1}_{4\text{x}4})\psi - e\overline{\psi}\gamma^\mu[\partial_\mu\beta(x)]\psi. \tag{3.12}$$

These observations together imply the simplest way to add a photon-spinor -interaction: in the form of $\mathcal{L}_I = ie\overline{\psi}\gamma^\mu A_\mu\psi$, where e is interpreted as the electric charge. We postulate that the local U(1) transformation that applies to ψ also applies to A_μ, with the transformation rule being (3.11). Hence, the full QED Lagrangian,

$$\mathcal{L}_{\text{QED}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}(i\gamma^\mu\partial_\mu - m\mathbb{1}_{4\text{x}4})\psi + ie\overline{\psi}\gamma^\mu A_\mu\psi, \tag{3.13}$$

transforms under this U(1) transformation as

$$\mathcal{L}_{\text{QED}} \to \mathcal{L}_{\text{QED}} + ie\overline{\psi}\gamma^\mu[\partial_\mu\alpha(x) + i\partial_\mu\beta(x)]\psi. \tag{3.14}$$

As the choice of $\alpha(x)$ is arbitrary, we can let $\alpha(x) = -i\beta(x)$, to ensure $\mathcal{L}_{\text{QED}}$ is invariant under this U(1) transformation.

Often, the interaction term of the QED Lagrangian is expressed together with the kinetic terms of the spinors, using a covariant derivative $D_\mu \equiv \partial_\mu + eA_\mu$:

$$\mathcal{L}_{\text{QED}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}(i\gamma^\mu D_\mu - m\mathbb{1}_{4\text{x}4})\psi. \tag{3.15}$$

The principle of gauge invariance

Here we introduced the photon first, for we knew that they do exist. Then we used more knowledge from classical electromagnetism, the transverse-polarization of light, to limit the degrees of freedom of our 4-vector field from 4 to 2, which matches that of Wigner's theorem and reality.

On the other hand, what if we didn't even know about the photon? If all we had was the Dirac Lagrangian, we could still reach the same interacting QED by looking at (3.12). We'd make the statement that the full theory should be invariant under this *gauge transformation*. We'd introduce some 4-vector field in an interaction term, to absorb it, i.e. the field should satisfy (3.11), and then reasonably demand that this 4-vector field should also have a free Lagrangian, (3.9), that is invariant under (3.11), associated to it.

This is the principle of *gauge invariance* at work: the dynamics of a physical theory follow from the demand that a Lagrangian is invariant under some local transformation. The unphysical degrees of freedom are handled by the *gauge fixing* procedure, which eqns. (3.4, 3.5) were an example of.

3.1.2 Yang-Mills theory

What if the local transformation of the previous section was not U(1), but a group with more structure? Consider $SU(N) \equiv \{U \in \mathrm{Mat}_{N\times N}(\mathbb{C}) : U^\dagger U = \mathbb{1}_{N\times N}, \mathrm{Det}(M) = 1\}$. Yang-Mills theory is based on these groups, and they form the foundation of the SM.

Under a general infinitesimal SU(N) -transformation, a set of N fields, $\phi \equiv (\phi_1, \phi_2, \ldots, \phi_N) \equiv \phi_i, i \in \{1, 2, ..., N\}$ transforms in the *fundamental representation* as [1, Chapter 25.1]

$$\phi_i \to \phi_i + i\alpha^a T^a_{ij}\phi_j, \tag{3.16}$$

where T^a are the traceless and hermitian generators of the group, with $a \in \{1, \ldots, \dim(SU(N))\} = \{1, \ldots, N^2 - 1\}$. The generators satisfy the mapping, also called the *Lie bracket*,

$$[T^a, T^b] = if^{abc}T, \tag{3.17}$$

where f^{abc} are the structure constants of the group.* We call the $f^{abc} = 0$ -groups *Abelian*, and the rest *non-Abelian*. In a matrix representation, the mapping (3.17) takes the form of a commutator $[T^a, T^b] = T^aT^b - T^bT^a$.

There is also the *adjoint representation*, under which the gauge fields of the theory transform.† The adjoint representation acts upon the vector space spanned by the

*Recall that for SU(2), $f^{abc} = \varepsilon^{abc}$, and $T^a = \tau^a = \frac{1}{2}\sigma^a$, with σ^a defined by (2.9). For SU(3), the generators are called the Gell-Mann matrices.

†See e.g. [1, Chapter 25.2.2] – the chapter on Wilson lines discusses a geometrical way of seeing gauge field theories.

generators. In other words, it has dim $N^2 - 1$; the matrices describing it are given by

$$(T^a_{\text{adj}})^{bc} = -if^{abc}. \tag{3.18}$$

These generators satisfy the same Lie bracket (3.17) as the generators of the fundamental representation. Other representations do exist, but these two are the most important for physics.

Generalizing the gauge field of QED to a non-Abelian gauge field of SU(N) goes as follows. Replace A_μ with A^a_μ and the field strength tensor $F_{\mu\nu}$ with

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - A^a_\mu \partial_\nu + gf^{abc} A^b_\mu A^c_\nu, \tag{3.19}$$

which is equivalent to replacing the covariant derivative with

$$(D_\mu)_{ij} = \delta_{ij}\partial_\mu - igA^a_\mu T^a_{ij}, \tag{3.20}$$

where indices i, j are the SU(N) matrix indices, μ, ν are Lorentz-indices, and a is again an SU(N) index that labels the generator. The covariant derivative and the field strength tensor are related by

$$-igF^a_{\mu\nu}T^a = [D_\mu, D_\nu]. \tag{3.21}$$

Finally, we may write the full massless* SU(N) invariant Lagrangian with spin 1 gauge fields and spin 1/2 spinor fields as

$$\mathcal{L}_{\text{Y-M}} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a + \overline{\psi}_i(\delta_{ij}\gamma^\mu\partial_\mu - ig\gamma^\mu A^a_\mu T^a_{ij})\psi_j. \tag{3.22}$$

Interpreting this as a generalized massless QED, this Lagrangian implies the existence of $N^2 - 1$ gauge particles, i.e. gauge bosons as their spin is 1, in interactions with at least N massless fermions that come with the Dirac Lagrangian.

Quantum field theoretically this interpretation is justified if we can quantize the theory. Everything we've done so far with the Yang-Mills theory has been classical, as no creation-annihilation operators have been mentioned. The gauge field theories are systems with constraints, which are more conveniently quantized with path integral methods [31, 32, 33]. These, however, are beyond the topic of this work despite their interesting and important nature.

*A discussion on the mass-terms is in chapter 3.2.5.

3.2 The electroweak theory

The SM of electroweak theory is based on the gauge group $SU(2)_L \times U(1)_Y$, where the subscripts stand for left-chiral and weak hypercharge, respectively.* On this group, the gauge-invariant Lagrangian for massless fields is formulated. The Higgs field is then used to generate mass-terms in a gauge-invariant way, and to induce a symmetry breaking that results in QED and weak interactions.

3.2.1 Electroweak gauge group

In the SM of electroweak interactions we introduce three generations of (fundamental representation) $\mathrm{SU}(2)_L$ doublets built from left-chiral quark (Q) and lepton (L) fields:

$$L_l \in \left\{ \begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix}, \begin{pmatrix} \nu_{\mu L} \\ \mu_L \end{pmatrix}, \begin{pmatrix} \nu_{\tau L} \\ \tau_L \end{pmatrix} \right\}, \qquad Q_i \in \left\{ \begin{pmatrix} u_L \\ d_L \end{pmatrix}, \begin{pmatrix} c_L \\ s_L \end{pmatrix}, \begin{pmatrix} t_L \\ b_L \end{pmatrix} \right\}, \tag{3.23}$$

where e, μ and τ refer to electron, muon and tau; and ν_l to their neutrino counterparts; $l \in \{e, \mu, \tau\}$, and $i \in \{1, 2, 3\}$. There are also their right-chiral counterparts, which are introduced as singlets under $\mathrm{SU}(2)_L$:

$$\begin{aligned} &l_R \in \{e_R, \mu_R, \tau_R\} \\ &u_{i,R} \in \{u_R, c_R, t_R\}, \qquad d_{i,R} \in \{d_R, s_R, b_R\}. \end{aligned} \tag{3.24}$$

Notably, the neutrino of the SM has no right-chiral realization.

We denote a general gauge transformation under $SU(2)_L \times U(1)_Y$ as

$$U_{\mathrm{EW}}(\theta_a, \eta) = \exp\Big(i[gT_a\theta_a + g'Y\eta]\Big), \tag{3.25}$$

where T_a are the three generators of $\mathrm{SU}(2)_L$ and Y is the generator of $\mathrm{U}(1)_Y$.† $\theta_a, \eta \in \mathbb{R}$ are parameters and g and g' are the coupling constants corresponding to SU(2) and U(1), respectively. Using this notation, the right-chiral lepton fields transform as

$$l_R \to U_{\mathrm{EW}}(\theta_a, \eta) l_R = \exp\Big(ig'Y\eta\Big) l_R, \tag{3.26}$$

i.e. they satisfy $T_a l_R = 0$, and the left-chiral fields as

$$L_l \to U_{\mathrm{EW}}(\theta_a, \eta) L_l = \exp\Big(i[gT_a\theta_a + g'Y\eta]\Big) L_l. \tag{3.27}$$

The transformations for quarks are similar.

*Sometimes this group is denoted $SU(2)_T \times U(1)_Y$, where the subscripts stand for weak isospin and hypercharge. The notation $SU(2)_W \times U(1)_Y$ is also seen, where W refers to the weak interactions.

†There exist two conventions on this. Either Y, or Y/2. In this work, we'll use the former.

Recall that a Lorentz-invariant mass-term for fermions requires a specific combination of L- and R-chiral fields. As we're building our model on the principle of gauge invariance in addition to Lorentz invariance, the transformations (3.26, 3.27) imply that a Lorentz-invariant massterm is not gauge invariant under the group $SU(2)_L \times U(1)_Y$:

$$\overline{L}_l l_R \to e^{-igT_a\theta_a}\overline{L}_l l_R. \tag{3.28}$$

Another complication of the above equation is the combinations $\nu_{lL} l_R$, which do not represent mass-terms, but interactions. We'll postpone the study of the fermionic mass-terms to chapter 3.2.5 and focus first on the gauge bosons.

3.2.2 The Brout-Englert-Higgs mechanism and EWSB

To introduce masses in a gauge-invariant way, the electroweak theory employs the Brout-Englert-Higgs (BEH) -mechanism, which also reproduces electromagnetism from the more general gauge group of $SU(2)_L \times U(1)_Y$ in a process called electroweak symmetry breaking (EWSB).

We define the Higgs doublet

$$H = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}, \tag{3.29}$$

where the superscripts denote charged (+) and neutral (0) fields and are justified once the full theory is explained. For our purposes we only need the neutral component ϕ^0.

H is defined to transform under the fundamental representation of $SU(2)_L$, and has hypercharge $Y = \frac{1}{2}$, i.e. $B_\mu Y H = \frac{1}{2} B_\mu H$. Under Lorentz transformations, H is a scalar. We introduce the Higgs into the theory with a Lagrangian

$$\begin{aligned}\mathcal{L}_H &\equiv (D_\mu H)^\dagger (D_\mu H) + m^2 H^\dagger H - \lambda (H^\dagger H)^2 \\ &= |D_\mu H|^2 - V(H),\end{aligned} \tag{3.30}$$

where the covariant derivative of the Higgs field is

$$D_\mu H = \partial_\mu H - igW_\mu^a T^a H - \frac{1}{2} ig' B_\mu H, \tag{3.31}$$

in which B_μ is the $U(1)_Y$ gauge boson and W_μ^a is the $SU(2)_L$ boson.

The potential $V(H)$ attains its minimum value at $|H| = \frac{v}{\sqrt{2}} = \sqrt{\frac{m^2}{2\lambda}}$. We call this the vacuum expectation value (vev) of the Higgs field, which in the doublet notation can be chosen to be

$$\langle H \rangle = \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix}. \tag{3.32}$$

We then consider the excitations of the Higgs field around this minimum:

$$H(x) = \exp\left(2i\frac{\pi^a(x)\sigma^a}{v}\right)\begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} + \frac{h(x)}{\sqrt{2}} \end{pmatrix}, \tag{3.33}$$

and immediately do a gauge-fix to simplify the calculations. The simplest gauge for our current purposes is the *unitarity gauge*, in which $\pi^a = 0$, or $H = H^\dagger$. Note that we could also fix the gauge before the vev is attained, and that this fixes only three of the four degrees of freedom in $SU(2)_L \times U(1)_Y$.

Inserting (3.33), the kinetic term of the Higgs Lagrangian (3.30) becomes [1, Chapter 29.1]

$$|D_\mu H|^2 = g^2\frac{v^2 + h(x)^2}{8}\left[(W_\mu^1)^2 + (W_\mu^2)^2 + \left(\frac{g'}{g}B_\mu - W_\mu^3\right)^2\right], \tag{3.34}$$

which represent mass-terms for three gauge bosons and also their interactions with the excitations $h(x)$ around the Higgs vacuum.

With this, we interpret the linear combination of B_μ and W_μ^3 as a massive gauge boson. To this end we introduce the *weak mixing angle* θ_w by

$$\tan\,\theta_w = \frac{g'}{g}, \tag{3.35}$$

and a rotation in the B_μ, W_μ^3 -plane with

$$\begin{aligned} Z_\mu &= \cos\,\theta_w W_\mu^3 - \sin\,\theta_w B_\mu, \\ A_\mu &= \sin\,\theta_w W_\mu^3 + \cos\,\theta_w B_\mu, \end{aligned} \tag{3.36}$$

which satisfies for example $\frac{1}{\cos\,\theta_w}Z_\mu = \frac{g'}{g}B_\mu - W_\mu^3$. The third massive gauge boson is therefore denoted Z_μ with mass $m_Z = \frac{1}{2\cos\,\theta_w}gv$. A massless gauge boson is then A_μ, which we interpret as the electromagnetic vector potential once we consider the neutral current (NC) interactions in the next section.

The remaining two gauge bosons, W_μ^1, W_μ^2, are similarly better understood by considering the charged current (CC) interactions, see sec. 3.2.4.

3.2.3 Neutral current (NC) interactions

Into the EW theory we include also the SU$(2)_L$ Yang-Mills -Lagrangian (3.22) and the $U(1)_Y$ -contributions:

$$\begin{aligned} \mathcal{L}_{\text{Y-M}} = &-\frac{1}{4}W_{\mu\nu}^a W_a^{\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} \\ &+ \overline{L}_l(\gamma^\mu\partial_\mu \mathbb{1} - ig\gamma^\mu W_\mu^a T^a - ig'\gamma^\mu B_\mu Y\mathbb{1})L_l \\ &+ \bar{l}_R(\gamma^\mu\partial_\mu - ig'\gamma^\mu B_\mu Y)l_R. \end{aligned} \tag{3.37}$$

Here we've omitted the quarks and left the $SU(2)_L$ matrix indices and a sum over lepton flavors, $\sum_l$, implicit. We denote $B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$, where B is the $U(1)_Y$ gauge boson. Similarly for $W^a_{\mu\nu}$ and the $\mathrm{SU}(2)_L$ bosons.

Making the doublet structure of the second line explicit, we retrieve the interactions between L-chiral fields and gauge bosons:

$$-i\begin{pmatrix}\overline{\nu}_{l,L} & \bar{l}_L\end{pmatrix}\gamma^\mu\begin{pmatrix}[g'B_\mu Y + \frac{g}{2}W^3_\mu] & \frac{g}{2}[W^1_\mu - iW^2_\mu] \\ \frac{g}{2}[W^1_\mu + iW^2_\mu] & [g'B_\mu Y - \frac{g}{2}W^3_\mu]\end{pmatrix}\begin{pmatrix}\nu_{l,L} \\ l_L\end{pmatrix}. \tag{3.38}$$

The diagonal terms yield

$$-i\overline{\nu}_{l,L}\gamma^\mu[g'B_\mu Y + \frac{g}{2}W^3_\mu]\nu_{l,L} - i\bar{l}_L\gamma^\mu[g'B_\mu Y - \frac{g}{2}W^3_\mu]l_L. \tag{3.39}$$

The first term corresponds to an interaction between neutrinos, latter to an interaction between charged leptons.

We know from QED that the interaction between charged leptons, such as electrons, features the massless electromagnetic vector potential as well. To match our current theory to that, we fix the hypercharge Y for the charged L-chiral leptons as $-\frac{1}{2}$, i.e. $Yl_L = -\frac{1}{2}l_L$. As the object that transforms under $U(1)_Y$ is the doublet L_l, we're actually enforcing $YL_l = -\frac{1}{2}L_l$.

Inserting this and applying (3.36) into (3.39) we retrieve

$$\mathcal{L}^{\mathrm{NC}}_L = \frac{-ig}{2\cos\theta_w}\Big\{\overline{\nu}_{l,L}\gamma^\mu Z_\mu \nu_{l,L} + (2\sin^2\theta_w - 1)\bar{l}_L\gamma^\mu Z_\mu l_L\Big\} + ig\sin\theta_w[\bar{l}_L\gamma^\mu A_\mu l_L] \tag{3.40}$$

for the L-chiral NC interactions*. The R-chiral terms contribute to the NC interactions as well:

$$\begin{aligned}-ig'\bar{l}_R\gamma^\mu B_\mu Y l_R &= -ig'\bar{l}_R\gamma^\mu(\cos\theta_w A_\mu - \sin\theta_w Z_\mu)Yl_R \\ &= -ig\sin\theta_w[\bar{l}_R\gamma^\mu A_\mu Y l_R] + i\frac{g}{2\cos\theta_w}2\sin^2\theta_w[\bar{l}_R\gamma^\mu Z_\mu Y l_R].\end{aligned} \tag{3.41}$$

As the interaction term in QED has both L- and R-chiral electrons with the same coefficient, we find from (3.40, 3.41) that $Yl_R = -1l_R$. Thus, the total NC Lagrangian is

$$\begin{aligned}\mathcal{L}^{\mathrm{NC}} = &\frac{-ig}{2\cos\theta_w}\Big\{\overline{\nu}_{l,L}\gamma^\mu Z_\mu\nu_{l,L} + (2\sin^2\theta_w - 1)\bar{l}_L\gamma^\mu Z_\mu l_L + 2\sin^2\theta_w\bar{l}_R Z_\mu\gamma^\mu l_R\Big\} \\ &+ ig\sin\theta_w[\bar{l}\gamma^\mu A_\mu l].\end{aligned} \tag{3.42}$$

*The term neutral current arises from the fact that (electric) charge conservation over the interaction implies the gauge bosons involved are charge-neutral.

From this, we read

$$g \sin\theta_w = g' \cos\theta_w = e, \tag{3.43}$$

where e is the coupling constant, electric charge, of QED. We also note that the charged L- and R-chiral leptons couple differently to the Z-boson. Weak interactions violate parity.

There exist some notable conventions for the notation. First is the notation which uses *currents* j_μ:

$$\mathcal{L}^{\mathrm{NC}} = \frac{-ig}{2\cos\theta_w} j_\mu^{\mathrm{Z}} Z^\mu - iej_\mu^{\mathrm{EM}} A^\mu, \tag{3.44}$$

and the other makes an explicit note on the vector-axial vector (V-A) nature of these interactions:

$$\begin{aligned} &\overline{\nu}_{l,L}\gamma^\mu \nu_{l,L} + (2\sin^2\theta_w - 1)\bar{l}_L\gamma^\mu l_L + 2\sin^2\theta_w \bar{l}_R\gamma^\mu l_R \\ &= \overline{\nu}_{l,L}\gamma^\mu(g_V^{\nu_l} - g_A^{\nu_l}\gamma^5)\nu_{l,L} + \bar{l}\gamma^\mu(g_V^l - g_A^l\gamma^5)l, \end{aligned} \tag{3.45}$$

where by appropriate choices of g_V, g_A, j_μ we match these to (3.42).

3.2.4 Charged current (CC) interactions and the gauge Lagrangian

We then turn towards the nondiagonal terms of (3.38):

$$-i\frac{g}{2}\Big\{\overline{\nu}_{l,L}\gamma^\mu[W_\mu^1 - iW_\mu^2]l_L + \bar{l}_L\gamma^\mu[W_\mu^1 + iW_\mu^2]\nu_{l_L}\Big\}. \tag{3.46}$$

These correspond to interactions between neutral neutrinos and their associated charged leptons. We're free to define the linear combinations in square brackets as the physical gauge bosons that participate in these interactions:

$$W_\mu^\pm \equiv \frac{1}{\sqrt{2}}(W_\mu^1 \mp iW_\mu^2). \tag{3.47}$$

It is found that the bosons $W^\pm$ have electrical charges of ± 1 in units of e.

With this, the charged current (CC) Lagrangian is

$$\begin{aligned} \mathcal{L}^{\mathrm{CC}} &= -i\frac{g}{\sqrt{2}}\Big\{\overline{\nu}_{l,L}\gamma^\mu W_\mu^+ l_L + \bar{l}_L\gamma^\mu W_\mu^- \nu_{l,L}\Big\} \\ &\equiv -i\frac{g}{2\sqrt{2}}\Big\{j_+^\mu W_\mu^+ + j_-^\mu W_\mu^-\Big\}. \end{aligned} \tag{3.48}$$

To conclude the discussion on the gauge interactions of the electroweak theory, we mention two points. First, the first two terms of $\mathcal{L}_{\text{Y-M}}$, eqn. (3.37), are affected by the rotation (3.36), as well as the linear combination (3.47). The result is rather lengthy, so we omit it from this text. It can be found in [1, Chapter 29.1] and [4, Chapter 3.6], for example. These terms contain the interactions between the gauge fields.

Second, there exists the Gell-Mann-Nishijima relation*

$$Q = T^3 + Y \tag{3.49}$$

between the electric charge Q and the weak interaction quantum numbers. We saw this relation in practise in section 3.2.3, where we had to fix couplings to the electromagnetic vector potential: for neutrinos we had $T^3 = +\frac{1}{2}$ and $T^3 = -\frac{1}{2}$ for charged leptons. Fixing $Y = -\frac{1}{2}$ for the doublet led to the neutrino decoupling from A, while reproducing the correct QED interaction term for the charged leptons. For the Higgs doublet (3.29), the choice $Y = +\frac{1}{2}$ leads to ϕ^+ being electrically charged while ϕ^0 is neutral.

3.2.5 Massive fermions

We're almost done with the SM electroweak theory. The last piece is the fermion masses. Recall from (3.28) that the gauge-invariant treatment has to be more involved than just a simple insertion of $\overline{\psi}_L \psi_R$. However, eventually we need to end up with this combination: we know this from QED.

Consider the Higgs-lepton Yukawa Lagrangian,

$$\mathcal{L}_{\mathrm{Y}} = -Y_{ll'} \overline{L}_l H l'_R + h.c., \tag{3.50}$$

where a sum $\sum_{l,l'}$, with $l, l' \in \{e, \mu, \tau\}$ is implicit, $Y_{ll'}$ is a 3 x 3 complex matrix of Yukawa couplings and $h.c.$ denotes the hermitian conjugate. This expression is invariant under $SU(2)_L \times U(1)_Y$, as we recall from earlier sections that $T^3 H = \frac{1}{2} H, T^3 \overline{L} = -\frac{1}{2}\overline{L}, Y l_R = -1, Y\overline{L} = \frac{1}{2}$, and $YH = \frac{1}{2}H$. Under Lorentz transformations, the R-chiral fields transform opposite to L-chiral fields, and the Higgs field is a scalar.

When we go through the usual steps of EWSB (sec. 3.2.2) and let the Higgs attain its vev (3.33), the above expression yields, in unitarity gauge,

$$\mathcal{L}_{\mathrm{Y}} = -\frac{v + h(x)}{\sqrt{2}} Y_{ll'} \bar{l}_L l'_R + h.c., \tag{3.51}$$

which we interpret as *almost* the Dirac masses for charged leptons, as well as their interactions with the excitations $h(x)$ around the Higgs vacuum. If these were to

*We remind that there exist two major conventions for this. In the other Y is replaced by $Y'/2$, in which Y' is called the weak hypercharge. To reproduce the same physics they must coincide, $Y = Y'/2$.

correspond to the physical particles, the matrix $Y_{ll'}$ would have to be diagonal, which it is has no reason to be.

Therefore we diagonalize it. Consider the above in matrix form:

$$\overline{\begin{pmatrix} e_L \\ \mu_L \\ \tau_L \end{pmatrix}} \begin{pmatrix} Y_{ee} & Y_{e\mu} & Y_{e\tau} \\ Y_{\mu e} & Y_{\mu\mu} & Y_{\mu\tau} \\ Y_{\tau e} & Y_{\tau\mu} & Y_{\tau\tau} \end{pmatrix} \begin{pmatrix} e_R \\ \mu_R \\ \tau_R \end{pmatrix}. \tag{3.52}$$

Apply the biunitary decomposition

$$Y = V_L D V_R^\dagger, \tag{3.53}$$

where V_L, V_R are unitary and D consists of real, positive-definite diagonal entries $D_{ij} = D_i \delta_{ij}$:

$$\begin{aligned} \overline{\begin{pmatrix} e_L \\ \mu_L \\ \tau_L \end{pmatrix}} \begin{pmatrix} Y_{ee} & Y_{e\mu} & Y_{e\tau} \\ Y_{\mu e} & Y_{\mu\mu} & Y_{\mu\tau} \\ Y_{\tau e} & Y_{\tau\mu} & Y_{\tau\tau} \end{pmatrix} \begin{pmatrix} e_R \\ \mu_R \\ \tau_R \end{pmatrix} &= \overline{V_L^\dagger \begin{pmatrix} e_L \\ \mu_L \\ \tau_L \end{pmatrix}} \begin{pmatrix} y_e & 0 & 0 \\ 0 & y_\mu & 0 \\ 0 & 0 & y_\tau \end{pmatrix} V_R^\dagger \begin{pmatrix} e_R \\ \mu_R \\ \tau_R \end{pmatrix} \\ &= \overline{\begin{pmatrix} e''_L \\ \mu''_L \\ \tau''_L \end{pmatrix}} \begin{pmatrix} y_e & 0 & 0 \\ 0 & y_\mu & 0 \\ 0 & 0 & y_\tau \end{pmatrix} \begin{pmatrix} e''_R \\ \mu''_R \\ \tau''_R \end{pmatrix}. \end{aligned} \tag{3.54}$$

This defines the double-primed fields as the physical, massive fields of the charged leptons. See for example [4, Chapter 4.1] for proof that this diagonalization is indeed possible.

With this, the Higgs-lepton Lagrangian becomes

$$\mathcal{L}_{\mathrm{Y}} = -\frac{v + h(x)}{\sqrt{2}} y_l \bar{l}''_L l''_R + h.c., \tag{3.55}$$

from which we read the masses of the physical leptons to be

$$m_l = \frac{v}{\sqrt{2}} y_l. \tag{3.56}$$

Note that the Yukawa couplings y_l can not be predicted from the theory alone.

As we've transformed the charged lepton fields, we need to check how this affects the other terms of the theory that feature them. The CC interactions (3.48) become

$$\overline{\nu}_{l,L} \gamma^\mu W_\mu^+ V_L^\dagger l''_L = \overline{\nu}''_{l,L} \gamma^\mu W_\mu^+ l''_L, \tag{3.57}$$

where we've transformed the massless SM neutrinos as

$$\begin{pmatrix} \nu''_{e,L} \\ \nu''_{\mu,L} \\ \nu''_{\tau,L} \end{pmatrix} = V_L \begin{pmatrix} \nu_{e,L} \\ \nu_{\mu,L} \\ \nu_{\tau,L} \end{pmatrix}. \tag{3.58}$$

These neutrinos are called the *flavor neutrinos*. And so, the CC interactions are unchanged *for the leptons.** With the neutrino, we make the same interpretation as with the charged leptons: the transformed set corresponds to what we eventually feature in the SM interactions.

The NC interactions (3.42) are unchanged under these transformations, for V_L and V_R are unitary.

From now, we'll drop the double-primed notation and understand that the Lagrangian corresponds to the fields that have been rotated with (3.54) and (3.58), once EWSB has been accounted for.

A recollection

We started from the full $SU(2)_L \times U(1)_Y$ gauge- and Lorentz-invariant electroweak Lagrangian

$$\mathcal{L}_{\text{EW}} = \mathcal{L}_H + \mathcal{L}_{\text{Y-M}} + \mathcal{L}_{\text{Y}}, \tag{3.59}$$

where $\mathcal{L}_H$ is given by (3.30), $\mathcal{L}_{\text{Y-M}}$ by (3.37), and $\mathcal{L}_{\text{Y}}$ by (3.50).

Going through the theoretical process of applying EWSB, followed by various field rotations and matching parameters to reproduce QED, we retrieved the appropriate mass-terms for fermions and three gauge fields as well as their interactions.

Quarks were mostly ignored, but their treatment in the EW theory is analogous to that of the leptons.

*For quarks, one finds the Cabibbo-Kobayashi-Maskawa (CKM) matrix.

4. The neutrino

The story of the neutrino, while strongly intertwined with the story of the EW theory [7, 13], does not end with the SM. With the observation of neutrino oscillations in 1998 [14], we can decisively declare the neutrino as a massive particle – something the SM does not consider.

In this section, the standard treatment for massive neutrinos is presented, following mostly [3, 4]. In addition, a short glance to the current experimental status is made, as well as some remarks to physics beyond the standard model (BSM).

4.1 The massive neutrino

The neutrino is a spin 1/2 -particle [7], and so the demand of Lorentz invariance restricts the phenomenologically possible mass terms to either Dirac- or Majorana-types, which were discussed in chapter 2.

Recall the CC and NC -interactions for the flavor neutrinos, discussed in chapter 3:

$$\mathcal{L}_{\nu}^{\mathrm{CC}} = -i\frac{g}{\sqrt{2}}\Big\{\overline{\nu}_{l,L}\gamma^{\mu}W_{\mu}^{+}l_L + \bar{l}_L\gamma^{\mu}W_{\mu}^{-}\nu_{l,L}\Big\}, \tag{4.1}$$

$$\mathcal{L}_{\nu}^{\mathrm{NC}} = -\frac{ig}{2\cos\theta_w}\Big\{\overline{\nu}_{l,L}\gamma^{\mu}Z_{\mu}\nu_{l,L}\Big\}. \tag{4.2}$$

In our notation, the neutrino fields $\nu_{l,L}$ have been once rotated to absorb the rotations of the charged L-chiral lepton fields. They are the massless *flavor neutrinos*, as discussed in chapter 3.2.5.

In the standard treatment, it is the flavor neutrino fields which enter the mass terms. There are three possibilities that are considered.

The Dirac mass term

Assuming the existence of 3 right-chiral neutrino fields $\nu_{l,R}$, which do not participate in the SM interactions and are sometimes called *sterile neutrinos* because of this, we may include a mass term similar to that of charged leptons:

$$\mathcal{L}^{\mathrm{D}} \equiv -\overline{\nu}_{l,L} M^{D}_{ll'} \nu_{l',R} + h.c., \tag{4.3}$$

where indices $l, l' \in \{e, \mu, \tau\}$, and M^D is a general 3 by 3 complex matrix like the Yukawa matrix of (3.51). A term like this follows from the BEH-mechanism just like the terms for charged leptons.

The Majorana mass term

Assuming that lepton number conservation can be violated, we may include a Majorana -type mass term, built from the flavor neutrinos of the SM:

$$\mathcal{L}^{\mathrm{M}} \equiv -\frac{1}{2}\overline{\nu}_{l,L} M^{M}_{l,l'} (\nu_{l',L})^c + h.c., \tag{4.4}$$

where M^M is a 3 by 3 complex matrix, and we've adopted the notation $\nu^c \equiv C\overline{\nu}^T$ as used in [3].* This agrees with the notation of chapter 2.7.

The Dirac-Majorana mass term

Assuming both the existence of 3 *sterile* R-chiral fields† $\nu_{l,R}$ and the violation of lepton number, we may include a mix of the above two cases:

$$\mathcal{L}^{\mathrm{D\text{-}M}} \equiv -\frac{1}{2}\overline{\nu}_{l,L} M^{ML}_{l,l'} (\nu_{l',L})^c - \overline{\nu}_{l,L} M^{D}_{l,l'} \nu_{l',R} - \frac{1}{2}\overline{\nu^c_{l,R}} M^{MR}_{l,l'} \nu_{l',R} + h.c., \tag{4.5}$$

where all matrices M are complex 3 by 3.

4.1.1 Diagonalizing the mass matrices

Similar to eqn. (3.51), the mass terms (4.3 - 4.5) do not represent massive fields until they've been diagonalized.

*When used with 4-component spinors, this is a common shorthand of the charge conjugation. In the context of Majorana neutrinos this is often encountered, but it should be interpreted more carefully [34]: the SM neutrino $\nu_{l,L}$ is not a 4-component spinor, it is L-chiral. Then, according to eqn. (2.73), $\nu^c_{l,L}$ is R-chiral. But charge conjugation is an internal symmetry, it should do nothing to the Lorentz group transformation properties of the field. This case illustrates the point made in the beginning of chapter 2.6, that those interpretations of the discrete transformations are sensible only in the context of Dirac 4-spinors.

†There are no special reasons to assume only 3 sterile neutrinos. This model can be generalized to contain more in a straightforward way. [3]

The Dirac mass term

The Dirac mass term is diagonalized in the same way as the matrix of Yukawa couplings of eqn. (3.51). We diagonalize with a biunitary transformation [4],

$$M^D = V_L^{D\dagger} m V_R^D, \tag{4.6}$$

where V_L^D, V_R^D are 3 by 3 unitary matrices (most commonly denoted with a letter U instead of V), $m_{\alpha\beta} = m_\alpha \delta_{\alpha\beta}$, and $m_\alpha > 0$. This results in

$$\mathcal{L}^D = -m_i \overline{\nu^D}_i \nu_i^D, \tag{4.7}$$

where a sum over i is understood, and $\nu_i^D = \nu_{i,L}^D + \nu_{i,R}^D$, in which

$$\nu_{i,L}^D \equiv V_L^{D\dagger} \begin{pmatrix} \nu_{e,L}, & \nu_{\mu,L}, & \nu_{\tau,L} \end{pmatrix}^T \tag{4.8}$$

$$\nu_{i,R}^D \equiv V_R^D \begin{pmatrix} \nu_{e,R}, & \nu_{\mu,R}, & \nu_{\tau,R} \end{pmatrix}^T \tag{4.9}$$

$$\nu_i^D \equiv \begin{pmatrix} \nu_1, & \nu_2, & \nu_3 \end{pmatrix}^T \tag{4.10}$$

The SM NC interaction (4.2) is invariant under the redefinition (4.8), for the transformation is unitary. The SM CC interaction (4.1), however, is not. It becomes

$$\mathcal{L}^{\text{CC}} = -i \frac{g}{\sqrt{2}} \Big\{ \overline{\nu_{i,L}^D} V_L^{D\dagger} \gamma^\mu W_\mu^+ l_L + h.c. \Big\}. \tag{4.11}$$

In this expression, we would be wise to not absorb the rotation $V_L^{D\dagger}$ to the charged lepton fields, as they are the L-chiral components of fields of definite masses, (3.55), and this property would be ruined if we rotated them again.

The massive Dirac neutrino thus enters the CC interaction in a mixed form, and the lepton flavor conservation is broken. The situation is analogous to that of quarks and their CKM matrix [3].

The Majorana mass term

The process for the Majorana mass term is the same as above, but this time the mass matrix is symmetric: recall the fermionic anticommutation relation for spin 1/2 fields (2.55) and the properties of C (2.74). These imply [3]

$$\overline{\nu}_{l,L} M_{l,l'}^M (\nu_{l,L})^c = \overline{\nu}_{l,L} [M_{l,l'}^M]^T (\nu_{l,L})^c. \tag{4.12}$$

Therefore, the diagonalization is achieved with

$$M^M = V^{M\dagger} m V^{M*}, \tag{4.13}$$

where $V_L^{M\dagger}$ is unitary 3 by 3 matrix, the $\dagger$ has been added for later convenience, $m_{\alpha\beta} = m_\alpha \delta_{\alpha\beta}$, and $m_\alpha > 0$. This results in

$$\mathcal{L}^M = -\frac{1}{2} m_i \overline{\nu^M}_i \nu_i^M, \tag{4.14}$$

where a sum over i is understood, and $\nu_i^M = \nu_{i,L}^M + (\nu_{i,L}^M)^c = (\nu_i^M)^c$, in which

$$\nu_{i,L}^M \equiv V^{M\dagger} \Big(\nu_{e,L}, \quad \nu_{\mu,L}, \quad \nu_{\tau,L}\Big)^T \tag{4.15}$$

$$\nu_i^M \equiv \Big(\nu_1, \quad \nu_2, \quad \nu_3\Big)^T \tag{4.16}$$

The property $\nu_i^M = (\nu_i^M)^c$ also implies $\nu_i = \nu_i^c$. The massive fields found by diagonalizing (4.4) are Majorana fields. Their associated particles are their own antiparticles.

Once again, the SM NC interaction (4.2) is invariant, but the CC interaction (4.1) is not:

$$\mathcal{L}^{\text{CC}} = -i \frac{g}{\sqrt{2}} \Big\{ \overline{\nu_{i,L}^M} V^{M\dagger} \gamma^\mu W_\mu^+ l_L + h.c. \Big\}. \tag{4.17}$$

The Dirac-Majorana mass term

The Dirac-Majorana mass term can be expressed as [3]

$$\mathcal{L}^{\text{D-M}} = -\frac{1}{2} \overline{n}_L M^{\text{D-M}} (n_L)^c + h.c., \tag{4.18}$$

where

$$n_L = \begin{pmatrix} \nu_L \\ (\nu_R)^c \end{pmatrix}, \qquad M^{\text{D-M}} = \begin{pmatrix} M_L^M & M^D \\ (M^D)^T & M_R^M \end{pmatrix}, \tag{4.19}$$

in which $M^{\text{D-M}}$ is a symmetrical 6 by 6 complex matrix. The diagonalization is then done with

$$M^{\text{D-M}} = V^{\text{D-M}} m (V^{\text{D-M}})^T, \tag{4.20}$$

in which $V^{\text{D-M}}$ is unitary 6 by 6 matrix, $m_{\alpha\beta} = m_\alpha \delta_{\alpha\beta}$, and $m_\alpha > 0$. This results in

$$\mathcal{L}^{\text{D-M}} = -\frac{1}{2} m_i \overline{\nu_i^{\text{D-M}}} \nu_i^{\text{D-M}}, \tag{4.21}$$

where a sum over i is understood, and $\nu_i^{\text{D-M}} = \nu_{i,L}^{\text{D-M}} + (\nu_{i,L}^{\text{D-M}})^c = (\nu_i^{\text{D-M}})^c$, in which

$$\nu_{i,L}^{\text{D-M}} \equiv V^{D-M} \Big(\nu_{e,L}, \quad \nu_{\mu,L}, \quad \nu_{\tau,L}, \quad (\nu_{e,R})^c, \quad (\nu_{\mu,R})^c, \quad (\nu_{\tau,R})^c\Big)^T \tag{4.22}$$

$$\nu_i^{\text{D-M}} \equiv \Big(\nu_1, \quad \cdots, \quad \nu_6\Big)^T \tag{4.23}$$

The property $\nu_i^{\text{D-M}} = (\nu_i^{\text{D-M}})^c$ again implies $\nu_i = (\nu_i)^c$. The massive fields found by diagonalizing (4.5) are Majorana fields.

The SM NC interaction (4.2) is invariant, but the CC interaction (4.1) is not. The flavor fields $\nu_{l,L}$ are a mixture of the 6 massive Majorana fields ν_i, instead of three fields as was the case for Dirac mass and Majorana mass terms.

4.1.2 The Weinberg operator

When introducing the Majorana-type mass terms in chapter 4.1, we did not discuss how such a term would be generated in a gauge-invariant way. Now we do.

There exists an extension to the SM called the 5-dimensional Weinberg operator [35]. It is the only gauge- and Lorentz invariant dim 5 operator that can be built from the SM fields. It is not renormalizable [3], and breaks lepton number conservation just like a Majorana -type mass term should.

We define the conjugated Higgs doublet,

$$\tilde{H} \equiv i\sigma_2 H^*, \tag{4.24}$$

and construct the $SU(2)_L \times U(1)_Y$ scalar $(\overline{L}_l \tilde{H})$, which yields

$$(\overline{L}_l \tilde{H}) \xrightarrow{\text{EWSB}} \frac{v + h(x)}{\sqrt{2}} \overline{\nu}_{l,L} \tag{4.25}$$

after the EWSB is applied. Hence, consider the Lagrangian

$$\mathcal{L}_{\text{d=5}} = -\frac{1}{\Lambda} \overline{L}_l \tilde{H} Y_{ll'} C (\overline{L}_{l'} \tilde{H})^T + h.c., \tag{4.26}$$

where C is the charge conjugation matrix, $Y_{ll'}$ are dimensionless constants, and Λ is a constant with dimension M^1. A sum over l and l' is implicit. Going through EWSB, we retrieve the mass Lagrangian as

$$\mathcal{L}_{\text{d=5}} \xrightarrow{\text{EWSB}} \mathcal{L}^M = -\frac{1}{2}(\frac{v^2}{\Lambda}) \overline{\nu}_{l,L} Y_{ll'} (\nu_{l',L})^c, \tag{4.27}$$

in which again a sum over l and l' is understood. This is exactly eqn. (4.4), with $M^M = \frac{v^2}{\Lambda} Y$.

An extremely tempting feature of this construction is the way the neutrino masses are suppressed. The mass terms (4.3 - 4.5) provide absolutely no explanation to the smallness of neutrino mass compared to that of other fermions, which depend on the Higgs vev and Yukawa couplings as $v^2 Y$. In the Weinberg Lagrangian, the constant Λ^{-1} provides the explanation. Λ is seen to characterize the scale at which the lepton number violating BSM physics becomes relevant. [4, 3]

4.1.3 The seesaw mechanism

The Weinberg operator can be seen as an *effective Lagrangian* generated from a higher-energy theory [3], similar to Fermi's famous 4-fermion theory [2]. We'll discuss three possibilities, all of which are called *seesaw mechanisms* due to the factor Λ of eqn.

(4.27) relating to a mass of some heavier particle in all of these constructions. As this mass increases, the mass of the neutrino decreases.

In this effective Lagrangian method, we introduce a set of new fields and a Lagrangian which leads to a tree-level interaction. From this, the mediating (heavy) particle is integrated out. The resulting term is the Weinberg operator, (4.26).

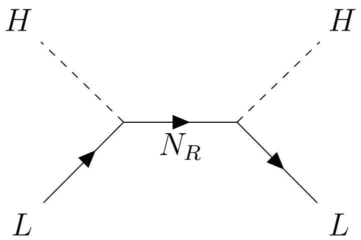
All of the three seesaws reproduce the same Weinberg operator with different factors Λ. They are further different in the interactions they predict as their complete Lagrangians differ. Such interactions can be used to discern which, if any, of the seesaws is realized in nature [19, 36]. From this text, we omit those phenomenological considerations, as our goal is to highlight methods of mass-generation for the neutrino.

Type I

The type I seesaw mechanism [3] assumes the existence of three heavy right-chiral fermions $N_{k,R}$ with the (BSM) Lagrangian featuring terms such as

$$\mathcal{L}^{\mathrm{I}} = -\overline{L}_l \overline{Y}_{lk} \tilde{H} N_{k,R} + \frac{1}{2} \overline{N_{k,R}^c} M_k N_{k,R} + h.c. \tag{4.28}$$

These terms define the tree-level interactions



from which we integrate out N_R (i.e. we consider energies $Q^2 \ll M_k^2$) and retrieve

$$\mathcal{L}_{\mathrm{eff}} = -\frac{1}{2} \Big(\overline{Y} \frac{1}{M} \overline{Y}^T \Big)_{ll'} \overline{L}_l \tilde{H} C (\overline{L}_{l'} \tilde{H})^T + h.c., \tag{4.29}$$

which corresponds to the Weinberg operator, with $\frac{1}{\Lambda} Y_{ll'}$ replaced by $\left(\overline{Y} \frac{1}{M} \overline{Y}^T \right)_{ll'}$. This has two implications. First, the higher the mass M_k, the smaller the L-chiral neutrino mass will be. Second, the mass M_k is related to the scale Λ at which the lepton number violating BSM physics start manifesting itself.

Type II

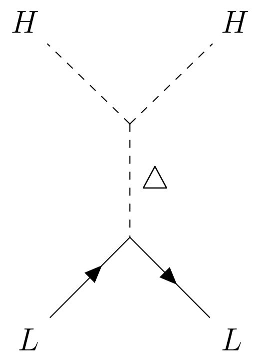
In type II [19], the fields introduced are three scalars in a $SU(2)_L$ triplet transforming in the adjoint representation. We denote the triplet as

$$\triangle = \begin{pmatrix} \triangle^- & -\sqrt{2}\triangle^0 \\ \sqrt{2}\triangle^{--} & -\triangle^- \end{pmatrix}, \tag{4.30}$$

and the corresponding extension to the SM Lagrangian features terms such as

$$\mathcal{L}^{\rm II} = \frac{1}{2}\overline{L}Y_\triangle i\sigma_2 L^c - m_\triangle^2 \mathrm{Tr}(\triangle^\dagger \triangle) - \lambda_\triangle H^T i\sigma_2\, \triangle^*\, H + h.c., \tag{4.31}$$

which define the tree-level interactions



from which we integrate out $\triangle$ and retrieve

$$\mathcal{L}_{\rm eff} = -\frac{\lambda_\triangle}{m_\triangle^2}(Y_\triangle)_{ll'}\overline{L}_l \tilde{H} C(\overline{L}_{l'}\tilde{H})^T + h.c., \tag{4.32}$$

which corresponds to the Weinberg operator, with $\frac{1}{\Lambda}Y_{ll'}$ replaced by $\frac{\lambda_\triangle}{m_\triangle^2}(Y_\triangle)_{ll'}$.

Type III

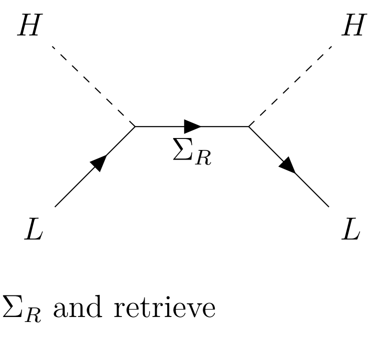
The type III seesaw [36] introduces three heavy fermionic triplets

$$\Sigma_R = \begin{pmatrix} \Sigma^0/\sqrt{2} & \Sigma^+ \\ \Sigma^- & -\Sigma^0/\sqrt{2} \end{pmatrix}, \tag{4.33}$$

with the corresponding extension to the SM Lagrangian having terms such as

$$\mathcal{L}^{\rm III} = \sqrt{2}\,\overline{L}\,Y_\Sigma\,\Sigma_R^c\,\tilde{H} + \frac{1}{2}\mathrm{Tr}(\overline{\Sigma_R}\,M_\Sigma\,\Sigma_R^c) + h.c. \tag{4.34}$$

These terms define the tree-level interactions



from which we integrate out Σ_R and retrieve

$$\mathcal{L}_{\rm eff} = -\frac{1}{2}\left(Y_\Sigma \frac{1}{M_\Sigma} Y_\Sigma^T\right)_{ll'} \overline{L}_l \tilde{H} C(\overline{L}_{l'}\tilde{H})^T + h.c., \tag{4.35}$$

which corresponds to the Weinberg operator, with $\frac{1}{\Lambda}Y_{ll'}$ replaced by $\left(Y_\Sigma \frac{1}{M_\Sigma} Y_\Sigma^T\right)_{ll'}$.

4.2 Neutrino mixing and oscillations

One possible theoretical consequence of a massive neutrino is the neutrino oscillation phenomenon, first predicted by Pontecorvo [5, 6], later observed by Kajita et al. [14] and in many other experiments thereafter. When introducing the mass terms, we omitted a relevant discussion on the mixing matrix, which we shall handle now, for it is necessary in the treatment of oscillations.

4.2.1 The PMNS matrix

The different mass models (Dirac (4.8), Majorana (4.22)) both lead to mixed flavor neutrinos:

$$\nu_{i,L}^{D} = V_L^{D\dagger}\nu_{l,L} \implies \nu_{l,L} = V_L^{D}\nu_{i,L}^{D}, \tag{4.36}$$

$$\nu_{i,L}^{M} = V^{M\dagger}\nu_{l,L} \implies \nu_{l,L} = V^{M}\nu_{i,L}^{M}, \tag{4.37}$$

in which $\nu_{l,L}$ denotes the flavor neutrinos of the SM, while $\nu_{i,L}^{X}$ denotes the massive neutrino. In both these cases, the unitary matrix V^X is called the PMNS (Pontecorvo–Maki–Nakagawa–Sakata) matrix, which contains all the information about neutrino flavor mixing, much like the CKM matrix for quarks. It is thus relevant to consider how many physical parameters are there in the PMNS matrix.

Let's first consider the Dirac case, V^D, more closely. For an n by n unitary matrix, we count $n(n-1)/2$ rotation angles and $n(n+1)/2$ phases, which makes 3 angles and 6 phases for $n = 3$. We may thus decompose $V^D = S(\beta)U^D S^\dagger(\alpha)$, in which the two phase matrices $S(x)_{jk} = e^{ix_j}\delta_{jk}$, with $\alpha_1 = 0, \alpha_{j\geq 2} \in \mathbb{R}$ and $\beta_j \in \mathbb{R}$, and U^D (see eqn. 4.39) is the matrix that contains the angles and one phase*.

We may absorb three of these phases, β_j, to the charged leptons† as can be seen from the CC current (4.11). The phases α_j may be absorbed into the Dirac neutrino fields themselves as their phase is arbitrary. Hence, the matrix V^D can be rephased into U^D which contains 4 physical parameters: 3 rotation angles and one phase.

The Majorana case is similar, but the Majorana condition fixes the phases of the Majorana fields: none of the phases α_j can be absorbed. Hence, V^M can be rephased into $U^D S(\alpha)$, which contains 6 physical parameters: 3 angles and 3 phases. Notably, the matrix U^D is the same as in the Dirac case, and the distinction between Dirac and Majorana PMNS matrices is in the Majorana phases $S(\alpha)$.

*For the n-dimensional case, U^D contains $(n-1)(n-2)/2$ phases.

†Rephasing the charged lepton array does not rotate it, so we don't ruin the diagonality of the mass terms in doing so.

The PMNS matrix is of crucial importance when considering CP invariance, for if the weak interactions are CP invariant, then

$$\mathcal{O}_{\mathrm{CP}}\mathcal{L}^{\mathrm{CC}}\mathcal{O}_{\mathrm{CP}}^{-1} = \mathcal{L}^{\mathrm{CC}} \tag{4.38}$$

has to hold, where $\mathcal{O}_{\mathrm{CP}}$ denotes the operation of CP conjugation and $\mathcal{L}^{\mathrm{CC}}$ is the CC Lagrangian. This condition yields conditions for the PMNS matrix *if* the CC interaction is CP invariant, but as the CP issues are not of much concern in this work, we'll refer the reader to [3] for further details.

The standard parametrization for the Majorana PMNS matrix is

$$V^M = U^D S(\alpha) = \begin{pmatrix} c_{13}c_{12} & c_{13}s_{12} & s_{13}e^{-i\delta} \\ -c_{23}s_{12} - s_{23}c_{12}s_{13}e^{i\delta} & c_{23}c_{12} - s_{23}s_{12}s_{13}e^{i\delta} & c_{13}s_{23} \\ s_{23}s_{12} - c_{23}c_{12}s_{13}e^{i\delta} & -s_{23}c_{12} - c_{23}s_{12}s_{13}e^{i\delta} & c_{13}c_{23} \end{pmatrix} \begin{pmatrix} e^{i\alpha_1} & 0 & 0 \\ 0 & e^{i\alpha_2} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{4.39}$$

in which $c_{jk} = \cos\theta_{jk}$ and $s_{jk} = \sin\theta_{jk}$. The parameters $\theta_{jk} \in [0, \frac{\pi}{2}]$ and $\delta \in [0, 2\pi]$ [17]. δ is also called the Dirac phase, sometimes the CP phase, perhaps since it is the only phase that appears in the Jarlskog invariant (see chapter 4.2.2) which is also a measure of CP violation [4].

4.2.2 The standard treatment of neutrino oscillations in vacuum

In the standard treatment of neutrino oscillations in vacuum [4], one assumes a neutrino created in a CC interaction process is described by the flavor state

$$|\nu_l\rangle = \sum_i V_{li}^* |\nu_i\rangle, \tag{4.40}$$

in which V is the PMNS matrix (4.39), the conjugate follows from the CC Lagrangian and $|\nu_i\rangle$ is a massive neutrino state. We're keeping with the notation where l, l' denote flavor neutrinos and i, j, k massive neutrinos. For simplicity, it is assumed that the massive states are orthonormal, for then

$$\langle\nu_j|\nu_k\rangle = \delta_{jk} \implies \langle\nu_l|\nu_{l'}\rangle = \delta_{ll'}. \tag{4.41}$$

The massive states are then eigenstates of some Hamiltonian, with eigenvalues $E_j = \sqrt{\mathbf{p}^2 + m_j^2}$, i.e.

$$H|\nu_j\rangle = E_j|\nu_j\rangle. \tag{4.42}$$

Thus they evolve in time as

$$|\nu_j(t)\rangle = e^{-iE_jt}|\nu_j\rangle, \tag{4.43}$$

which in turn leads to the time-evolution of the flavor state:

$$|\nu_l(t)\rangle = \sum_i e^{-iE_it}V_{li}^*|\nu_i\rangle, \qquad |\nu_l(0)\rangle = |\nu_l\rangle. \tag{4.44}$$

Inverting the definition (4.40) yields

$$|\nu_i\rangle = \sum_l V_{li}|\nu_l\rangle, \tag{4.45}$$

which we insert to (4.44) to retrieve

$$|\nu_l(t)\rangle = \sum_{l'=e,\mu,\tau}\Big(\sum_i e^{-iE_it}V_{li}^*V_{l'i}\Big)|\nu_{l'}\rangle \tag{4.46}$$

as the flavor-state at time t. From this, the transition amplitude $l \to l'$ is read as

$$\mathcal{A}_{l\to l'}(t) = \langle\nu_{l'}|\nu_l(t)\rangle = \sum_i e^{-iE_it}V_{li}^*V_{l'i}, \tag{4.47}$$

which gives the transition probability as

$$P_{\nu_l\to\nu_{l'}} = |\mathcal{A}_{l\to l'}(t)|^2 = \sum_{i,j} V_{li}^*V_{l'i}V_{lj}V_{l'j}^*e^{-i(E_i-E_j)t}. \tag{4.48}$$

The ultrarelativistic assumption is then made:

$$E_i \approx E + \frac{m_i^2}{2E}, \qquad E = |\mathbf{p}| \tag{4.49}$$

which gives

$$E_i - E_j = \frac{\Delta m_{ij}^2}{2E}, \qquad \Delta m_{ij}^2 = m_i^2 - m_j^2. \tag{4.50}$$

Finally, we note that in neutrino experiments the propagation time is most often unknown. Instead we know the source-detector -distance L, and at the ultrarelativistic limit we approximate the propagation time as

$$t \approx L. \tag{4.51}$$

Which finalizes the standard oscillation treatment. The result is

$$P_{\nu_l\to\nu_{l'}} = |\mathcal{A}_{l\to l'}(t)|^2 = \sum_{i,j} V_{li}^*V_{l'i}V_{lj}V_{l'j}^*\mathrm{exp}\Big(-i\frac{\Delta m_{ij}^2 L}{2E}\Big). \tag{4.52}$$

We make some concluding remarks. First, the object $V_{li}^* V_{l'i} V_{lj} V_{l'j}^*$ (also known as the leptonic Jarlskog invariant) does not depend on Majorana phases:

$$V_{li}^* V_{l'i} V_{lj} V_{l'j}^* = e^{-i\alpha_i} U_{li}^* U_{l'i} e^{i\alpha_i} U_{lj} e^{i\alpha_j} e^{-i\alpha_j} U_{l'j}^* = U_{li}^* U_{l'i} U_{lj} U_{l'j}^*. \tag{4.53}$$

Hence, these transition probabilities are not suitable for differentiating between Dirac and Majorana neutrinos. Second, only a squared mass-difference Δm_{ij}^2 enters the transition probability (4.52) instead of individual neutrino masses. Third, the oscillation amplitude is determined solely by the combination (4.53), which is also a measure of CP violation.

Lastly, we remark that the neutrino oscillations inside matter (such as the sun) will not be treated in this work. The interested reader is referred to [3, 4, 37]. In short, the idea is that a neutrino in ordinary matter encounters electrons, but not muons and taus. Hence only ν_e can experience a CC interaction in such matter, which modifies the transition probability for ν_e compared to that of ν_τ or ν_μ.

4.3 BSM physics and the neutrino

While the SM has been a magnificent triumph, nowadays we understand it as an effective low-energy theory of something more fundamental. In this work we've studied the seesaw mechanism, which is an example of this way of thinking.

This thinking can be justified by the many problems of the SM. Besides the absence of mass terms for neutrinos, the SM does not contain dark matter (DM). The SM can not explain the baryon asymmetry of the universe (BAU). The SM has a hierarchy problem.* The Yukawa couplings related to masses of particles differ at 6 orders of magnitude between generations for no clear reason. [38]

The neutrinos may hold the answers to some of these shortcomings of the SM. Possible DM candidates include the massive sterile neutrinos [39, 40]. Neutrinos may also be the explanation for BAU [40], and they may explain the origin of the electroweak scale [41]. A recent paper suggests that a type I seesaw -model can explain simultaneously the BAU, EW scale, and the massive neutrino [42], though it should be stressed that the type I sterile neutrino is notoriously difficult to observe.

On the other hand, the massive neutrino has implications for cosmology [4, 43], which is an important avenue for the evaluation of the different neutrino models. The relic

*In the hierarchy problem, also known as fine-tuning problem, the loop corrections to the SM Higgs mass are proportional to the energy scale at which the SM is expected to break down. However, the SM Higgs can be much lighter than this scale, which necessitates fine-tuning of the parameters to match the Higgs mass with observations.

neutrinos of the cosmic neutrino background can be a window to the early universe [44].

4.3.1 Current experimental understanding of the neutrino

To conclude our study on the neutrino, we make a short detour to the latest Particle Data Group (PDG) review (chapters 14 and 26 of [17]), to recollect some of the current knowledge on neutrino measurements.

The experiments on neutrino oscillations* set the 3-neutrino -model described by the 3 by 3 PMNS matrix as the current paradigm, but there exist some anomalies that can not be explained with only 3 neutrinos. Under the 3-ν scheme, the masses can be in two possible orders: the normal ordering (NO) $m_1 < m_2 < m_3$ or the inverted ordering (IO) $m_3 < m_1 < m_2$. The mass-squared differences (4.50) satisfy $\Delta m_{21}^2 \ll |\Delta m_{31}^2|^2 \approx |\Delta m_{32}^2|^2$. Further, the neutrino mass orderings can be classified as

- Normal Hierarchical Spectrum (NH): $m_1 \ll m_2 < m_3$:
  $m_2 \approx \sqrt{\Delta m_{21}^2} \sim 8.6 \times 10^{-3}\text{eV}, m_3 \approx \sqrt{\Delta m_{32}^2 + \Delta m_{21}^2} \sim 0.05\text{eV},$
- Inverted Hierarchical Spectrum (IH): $m_3 \ll m_1 < m_2$:
  $m_1 \approx \sqrt{|\Delta m_{32}^2 + \Delta m_{21}^2|} \sim 0.0492\text{eV}, m_2 \approx \sqrt{|\Delta m_{32}^2|} \sim 0.05\text{eV},$
- Quasidegenerate Spectrum (QD): $m_1 \approx m_2 \approx m_3 \gg \sqrt{|\Delta m_{32}^2|}$.

As noted in section 4.2.2, there are six parameters to be measured in the 3-ν oscillation analysis: two mass-differences, 3 mixing angles and a phase which we also call the CP phase. The review gives an impressive list of Homestake, SAGE, KALLEX, GNO, Kamiokande, Super-Kamiokande, SNO, KamLAND, Daya-Bay, Reno, D-Chooz, SK, IC-DC, K2K, MINOS, T2K and NOνA as the current experiments that contribute to their measurement, and presents a comprehensive table of their measured values (Table 14.7 of [17]), which is rather lengthy to be included in this work. In all of the analyses, the best fit is for NO.

As a remarkably important result of recent times, we highlight the nonzero value of θ_{13}, a mixing matrix parameter which was assumed zero until 2012.

Oscillation experiments do not yield information on the mass-scales m_i, nor their nature as Dirac or Majorana particles. The PDG review lists some laboratory probes that are capable of these, which we now discuss.

*The experiments consist of solar, atmospheric, accelerator, and reactor -setups.

Kinematics of weak decays

The only source of model-independent information on neutrino masses (instead of mass-squared differences) is energy-momentum conservation in reactions where neutrinos, including possible non-relativistic neutrinos predicted by the tail of the β -decay spectrum, are present.

The most recent ^{3}H beta decay -result in the review is cited to be from KATRIN [45], which puts a 90% confidence level (CL) upper limit $m_{\nu_e}^{\text{eff}} < 1.1$ eV, where $m_{\nu_e}^{\text{eff}} = \sqrt{\sum_i m_i^2 |V_{ei}|^2}$. An alternative isotope for experiments is the electron-capture -decaying ^{163}Ho, which is being experimented at three different projects ECHo, HOLMES and NuMECS [46, 47, 48].

The effective masses of other flavors are listed as $m_{\nu_\mu} < 190$ keV (90% CL), $m_{\nu_\tau} < 18.2$ MeV (95% CL).

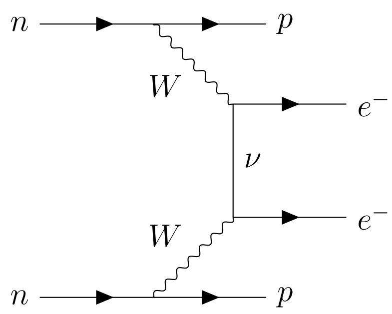
Putting together the information on mass differences, mixing matrix elements from oscillation experiments, and the upper bound of $m_{\nu_e}^{\text{eff}}$, the review describes a corresponding range for the mass of the lightest neutrino under the two possible orderings. In particular, currently we have the 95% CL estimate of $m_{\nu_e}^{\text{eff}} > 0.048$ (0.0085) eV for IO (NO).

$0\nu\beta\beta$ -decay

The neutrinoless double beta decay process

$$(A, Z) \to (A, Z + 2) + e^- + e^-$$

can be described by the diagram



that is only possible for Majorana neutrinos [49]. Some other new physics can also be the source of such phenomenon, but we shall not consider those cases here [17]. In any case the $0\nu\beta\beta$ -decay is a very hot topic for it promises an avenue of distinguishing whether neutrinos are Majorana or Dirac. A Google Scholar search yields over 20 publications and preprints under this topic just from year 2020.*

The PDG review [17] lists the ongoing experiments GERDA, KamLAND-Zen, EXO-200, SNO+, NEXT-White, CUORE, CUPID-0, AMoRE-Pilot, and NEMO-3 which are working with $0\nu\beta\beta$. The current upper bound for the half-life of the decay comes from KamLAND-Zen, which sets the effective Majorana mass at $m_{ee} = |\sum_i m_i V_{ei}^2| < 61-165$ meV. (Its lower bound is estimated from oscillation experiments at 0.016 eV at 95% CL for the IO.)

Cosmology

In addition to the abovementioned, the PDG review discusses cosmological constraints on neutrino parameters (chapter 26 of [17]). In addition to the data already discussed, cosmology provides an upper bound to the total mass of all neutrinos in the range $\sum m_\nu < 0.11 - 0.515$ eV (95% CL), depending on the cosmological model [55, 56].

Of purely cosmological interest, the review mentions that the relic neutrino background (also known as cosmic neutrino background, CνB), a prediction of the standard hot big bang -model, has been indirectly confirmed by now by the agreement of predictions and observations of: a) the primordial abundance of light elements; b) the cosmic microwave background (CMB); and c) the large-scale clustering of cosmological structures.

*To list just a few, to back up the claim about topic hotness: [50, 51, 52, 53, 54].

5. The method of unitarily inequivalent representations

The method of unitarily inequivalent representations, in short, concerns the diagonalization of a Hamiltonian. Examples of this method can be found in the Nambu-Jona-Lasinio -model [23]†, Bogoliubov's take on BCS theory [22], Haag's theorem [57], and the textbooks [58, 59, 60].

In this chapter, the principles of this method are applied to the Majorana neutrino in a manner outlined in references [21, 61]. The standard neutrino oscillation probability (4.52) is reproduced as the ultrarelativistic limit of a more general result. The formalism avoids the ill-defined flavor states, which can not be quanta of the flavor fields [62].

First we consider a two-flavor Majorana Lagrangian, and second a simple type I seesaw Lagrangian. We follow closely the reference article [21] which treats the two-flavor Dirac Lagrangian. We change our terminology and notation to match the reference article. The original work behind the discussion of this chapter is presented with more detail in appendix B.

5.1 A consistent model of the massive Majorana neutrino

Our starting point is the SM *flavor* neutrino. It is massless and has left helicity. As such, it is described by a L-chiral Weyl field, which we can express as a L-chiral projection (2.37) of a 4-component Dirac field (2.68), in which the spinors $u_\lambda(\mathbf{p})$ (A.3)

†The NJL model also happens to be one that popularized the concept of SSB in elementary particle physics before the SM of EW theory was introduced, being an important predecessor to the SM.

and $v_\lambda(\mathbf{p})$ (A.4) are taken at the massless limit. In other words, we write

$$\begin{aligned}\psi_{\nu_l,L} &\equiv \lim_{m\to 0} P_L \psi_{\nu_l}(\mathbf{x},t) \\ &= \lim_{m\to 0} P_L \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_\lambda \Big(a_{l\lambda}(\mathbf{p})u_\lambda(\mathbf{p})e^{-ipx} + b^\dagger_{l\lambda}(\mathbf{p})v_\lambda(\mathbf{p})e^{ipx}\Big), \end{aligned} \tag{5.1}$$

where we changed our notation from that of chapters 2 and 3 to that of [21]. The embedded quantum operators satisfy the canonical commutation relations (CCR) (2.55 - 2.57) and they are related to a vacuum state, which we denote as $|0\rangle$, s.t. $a_{l\lambda}(\mathbf{p})|0\rangle = b_{l\lambda}(\mathbf{p})|0\rangle = 0$.

The SM neutrino satisfies the Lagrangian

$$\mathcal{L}_{\nu\mathrm{SM}} = i\sum_l \left\{\psi^\dagger_{\nu_l,L}\overline{\sigma}_\mu\partial^\mu\psi_{\nu_l,L}\right\} \tag{5.2}$$

(as well as the couplings to the gauge bosons $W^\pm$ and Z (4.1, 4.2), which are not relevant to the case at hand.) To this Lagrangian we introduce a non-diagonal mass term, i.e. an interaction.

Let the interaction be a Majorana -type mass term (2.82) that features flavor mixing (4.4):

$$\begin{aligned}\mathcal{L}_{\mathrm{BSM}} &= i\sum_l \left\{\Psi^\dagger_{\nu_l,L}\overline{\sigma}_\mu\partial^\mu\Psi_{\nu_l,L}\right\} + \frac{i}{2}\sum_{ll'} m_{ll'}[\Psi^\dagger_{\nu_l,L}\sigma_2\Psi^*_{\nu_{l'},L} - \Psi^T_{\nu_l,L}\sigma_2\Psi_{\nu_{l'},L}] \\ &= \mathcal{L}_0 + \mathcal{L}_{\mathrm{mass}}. \end{aligned} \tag{5.3}$$

In this the *flavor* fields $\psi_{\nu_l,L}$ have been replaced by the *mixed* neutrino fields Ψ_{ν_l}, for this Lagrangian no longer describes the SM *flavor* neutrinos. Note that we need, or provide, no information on the origin of the mixing term, but it is understood that this is the Lagrangian we find after applying EWSB to some SM extension.

The only assumption we make for the field Ψ_{ν_l} is as follows. Let there exist a time when the interaction $\mathcal{L}_{\mathrm{mass}}$ is turned off, $t = 0$. At this time, the flavor fields are equivalent to the mixed fields:

$$\psi_{\nu_l,L}(\mathbf{x},0) = \Psi_{\nu_l}(\mathbf{x},0). \tag{5.4}$$

From this point onwards, we proceed with the usual canonical quantization scheme, see section 2.5.2, with the goal of expressing the Hamiltonian corresponding to (5.3) with appropriate quantum operators and then diagonalizing it. For simplicity, we consider a two-flavor case, such that

$$M = \begin{pmatrix} m_{ee} & m_{e\mu} \\ m_{e\mu} & m_{\mu\mu}\end{pmatrix}. \tag{5.5}$$

What follows is explained and calculated in detail in Appendix B.1. The diagonalized Hamiltonian is found to be

$$H = \int \frac{d^3p}{(2\pi)^3} \sum_{\lambda,i} \Omega_{ip} A^\dagger_{i\lambda}(\mathbf{p}) A_{i\lambda}(\mathbf{p}), \tag{5.6}$$

where $\Omega_{ip} = \sqrt{p^2 + m_i^2}$ and $A_{i\lambda}$ (B.28) are operators that satisfy the CCR. This corresponds to two free and massive Majorana fields with masses m_1 and m_2. We call these fields the *massive* neutrino fields $\Psi_i(\mathbf{x}, 0)$, in which $i = 1, 2$. The time-dependence of the fields Ψ_i is restored in the Heisenberg picture, so that

$$\Psi_i(\mathbf{x}, t) = e^{iHt} \Psi_i(\mathbf{x}, 0) e^{-iHt}, \tag{5.7}$$

in which H is given by eqn. (5.6). The operators then evolve as

$$A_{i\lambda}(\mathbf{p}, t) = A_{i\lambda}(\mathbf{p}, 0) e^{-i\Omega_{ip}t}. \tag{5.8}$$

As in section 2.5.2, we introduce a corresponding vacuum state, which we denote with $|\Phi_0\rangle$, and define through $A_{i\lambda}(\mathbf{p})|\Phi_0\rangle = 0$. We find, see (B.32), that this vacuum can be related to the vacuum $|0\rangle$ via

$$|\Phi_0\rangle = \prod_{i,p,\lambda} \Big(\alpha_{ip} - \beta_{ip} c^\dagger_{i\lambda}(\mathbf{p}) c^\dagger_{i\lambda}(-\mathbf{p}) \Big) |0\rangle, \tag{5.9}$$

where $c_{i\lambda}$ (B.26) are the rotated quantum operators related to the flavor field $\psi_{\nu_l,L}$. This result leads to $\langle 0|\Phi_0\rangle$ vanishing in the infinite momentum limit, see (B.35). Therefore, the vacua are orthogonal, and so are their corresponding Fock spaces.

We interpret the physical vacuum as $|\Phi_0\rangle$, for this is the vacuum that corresponds to the diagonalized and normal-ordered Hamiltonian that corresponds to a massive neutrino.

The infinite product of massless neutrino operators seen in (5.9) deserves further attention. This structure appears as we consider the operators of the massive (physical) particles in terms of the operators of the massless particles, and merely serves to show that there exists a vacuum annihilated by $A_{i\lambda}$ and that it is orthogonal to $|0\rangle$. The flavor number violating structure is to be expected from a model which has flavor number violation built into it, as per Coleman's theorem [63].

In this model, the flavor neutrino is a massless particle, hence not physical. The CC and NC interactions of the SM are seen as effective descriptions of the massive neutrinos, not massless neutrinos, interacting with other massive particles. Thus, the infinite product of flavor neutrinos does not imply a vacuum full of real particles.

To conclude, we would like to compare this vacuum with the BCS-Bogoliubov -model [22], where a similar vacuum structure* is interpreted as Cooper pairs of fundamental electrons. The BCS theory features physical particles (the electrons in a conductor) even before the diagonalization of its Hamiltonian. Such a process of "clothing", where an interaction transforms a "bare" (initial) state to a "dressed" (physical) state, changes the energy of the quantum in the case of BCS, and the mass in the relativistic case, such as the model we've treated, for example. [58, Chapter 12.5]

5.2 Applied to the seesaw mechanism

Next, we apply the same method to the one-flavor type I seesaw Lagrangian, which is [3]

$$\begin{aligned}\mathcal{L}_{\text{BSM}} = & \, i\Psi_L^\dagger\overline{\sigma}_\mu\partial^\mu\Psi_L + i\Psi_R^\dagger\sigma_\mu\partial^\mu\Psi_R \\ & + \frac{1}{2}m_D[\Psi_L^\dagger\Psi_R - \Psi_L^T\Psi_R^* + h.c.] \\ & + \frac{i}{2}m_R[\Psi_R^T\sigma_2\Psi_R - \Psi_R^\dagger\sigma_2\Psi_R^*] \\ = & \, \mathcal{L}_0 + \mathcal{L}_{\text{mass}}.\end{aligned} \tag{5.10}$$

We again assume there exists a time when the interaction is turned off, and the mixed fields $\Psi_{\nu,L}$ and $\Psi_{\nu,R}$ are equivalent to massless fields

$$\begin{aligned}\Psi_{\nu,L}(\mathbf{x},0) &= \psi_{\nu,L}(\mathbf{x},0), \\ \Psi_{\nu,R}(\mathbf{x},0) &= \psi_{\nu,R}(\mathbf{x},0).\end{aligned} \tag{5.11}$$

The field $\psi_{\nu,L}$ is identified with the SM flavor neutrino, and the field $\psi_{\nu,R}$ as some right-chiral fermion which extends the SM, but has yet to obtain its mass. The mode expansions are given by equations (B.3) and (B.40).

The analysis follows identical steps to that of the two-flavor Majorana case and is treated in detail in Appendix B.2. The diagonalized Hamiltonian again is found to satisfy

$$H = \int \frac{d^3p}{(2\pi)^3}\sum_{\lambda,i}\Omega_{ip}A_{i\lambda}^\dagger(\mathbf{p})A_{i\lambda}(\mathbf{p}), \tag{5.12}$$

but this time the operators $c_{i\lambda}$ that enter the definition of operators $A_{i\lambda}$, (B.28), are defined by (B.57), and the masses m_i that enter $\Omega_{ip} = \sqrt{p^2 + m_i^2}$ are defined by (B.56). The conclusions about time evolution and vacua, as discussed in the previous section, are exactly the same for this case.

*Also called the BCS ground state in literature.

We remark that the canonical seesaw mechanism assumes $m_R \gg m_D$. Under this assumption, we find the angle (B.54) $\theta \approx \frac{m_D}{m_R} \ll 1$, which gives the masses of the particles as $m_1 \approx m_R$ and $m_2 \approx m_D^2/m_R$.

5.3 Oscillating neutrino states

Now that we've constructed the massive neutrino Hamiltonian with the canonical quantization scheme and discussed the vacua, we see that constructing the oscillating neutrino states, (4.40), as

$$|\nu_l\rangle \sim \sum_i |\nu_i\rangle \tag{5.13}$$

is ambiguous. What is the vacuum upon which the states are defined? If this means

$$\sum_i |\nu_i\rangle = \sum_i A_i^\dagger |0\rangle, \tag{5.14}$$

then it certainly is ill-defined, for this is not the physical vacuum of this theory.* A step to a better direction is

$$|\nu_l\rangle \sim \sum_i |\nu_i\rangle = \sum_i A_i^\dagger |\Phi_0\rangle, \tag{5.15}$$

but it turns out that we need not impose any additional definitions like this.

In this model we formalize the oscillating neutrino states as the states created by acting on the physical vacuum with the flavor neutrino operators. Their action on $|\Phi_0\rangle$ shall be defined through the operators $A_{i\lambda}(\mathbf{p})$, by inverting the Bogoliubov transformation (B.28), which yields (B.38) in the 2-flavor Majorana case and (B.59) for the simple seesaw.

Consider now the two-flavor Majorana, and return to the seesaw later. We define the oscillating neutrino states via

$$|\nu_{l\lambda}(\mathbf{p})\rangle \equiv a_{l\lambda}^\dagger(\mathbf{p})|\Phi_0\rangle, \tag{5.16}$$

which yields

$$\begin{aligned} |\nu_{e\downarrow}(\mathbf{p})\rangle &= \alpha_{1p}\cos\theta A_{1\downarrow}^\dagger(\mathbf{p})|\Phi_0\rangle + \alpha_{2p}\sin\theta A_{2\downarrow}^\dagger(\mathbf{p})|\Phi_0\rangle, \\ |\nu_{\mu\downarrow}(\mathbf{p})\rangle &= \alpha_{1p}\cos\theta A_{1\downarrow}^\dagger(\mathbf{p})|\Phi_0\rangle + \alpha_{2p}\sin\theta A_{2\downarrow}^\dagger(\mathbf{p})|\Phi_0\rangle. \end{aligned} \tag{5.17}$$

*In this context, it must be mentioned that there has been a discussion for and against the flavor Fock space, i.e. the collection of states built using the vacuum $|0\rangle$, in the literature: see for example the introduction of [64] and references therein, as well as the recent comments [65, 66].

Interestingly these states are not orthogonal: $\langle\nu_{\mu\downarrow}(\mathbf{p})|\nu_{e\downarrow}(\mathbf{p})\rangle = \frac{1}{2}\sin(2\theta)\Big[\alpha_{2p}^2 - \alpha_{1p}^2\Big] \neq 0$. This differs from the standard formula (4.40). However, the requirement for orthogonality holds only for the massive neutrino states $A^\dagger_{i\lambda}|\Phi_0\rangle$, not for the massless flavor neutrinos [21]. (Note that in the case where $m_i \to 0$, we find $\langle\nu_{\mu\downarrow}(\mathbf{p})|\nu_{e\downarrow}(\mathbf{p})\rangle = 0$.)

From this, we retrieve the amplitude for an electron neutrino to oscillate into a muon neutrino after time t as

$$\langle\nu_{\mu\downarrow}(\mathbf{p})|e^{-iHt}|\nu_{e\downarrow}(\mathbf{p})\rangle = \frac{1}{2}\sin(2\theta)\Big[\alpha_{2p}^2 e^{-i\Omega_{2p}t} - \alpha_{1p}^2 e^{-i\Omega_{1p}t}\Big], \tag{5.18}$$

which as a formula is valid for all momenta and mass parameters, and is indeed the same form as found in [21], which considers the case of two flavors of Dirac neutrinos. This agrees with the standard treatment, in which oscillations can not distinguish between Dirac and Majorana neutrinos, as discussed in chapter 4.2.2.

From the amplitude (5.18), we find in the ultrarelativistic limit the usual oscillation probability for the 2-flavor case [3]

$$|\langle\nu_{\mu\downarrow}(\mathbf{p})|e^{-iHt}|\nu_{e\downarrow}(\mathbf{p})\rangle|^2 = \sin^2(2\theta)\,\sin^2\Big(\frac{m_2^2 - m_1^2}{4E}L\Big). \tag{5.19}$$

Finally, as another sanity check to the model, note that the vacuum oscillation amplitude for a L-helicity neutrino to end up as an R-helicity neutrino is found to be zero.

The case for the simple seesaw is indeed similar, but the interpretation and result is arguably more interesting. Due to the similarity of the two cases, we may read the seesaw results by the substitution $|\nu_{e\downarrow}(\mathbf{p})\rangle \to |\nu_{R\downarrow}(\mathbf{p})\rangle$ and $|\nu_{\mu\downarrow}(\mathbf{p})\rangle \to |\nu_{L\downarrow}(\mathbf{p})\rangle$ to eqn. (5.19). This gives us the probability for a SM flavor neutrino to oscillate into its BSM Majorana fermion counterpart at ultrarelativistic velocities:

$$\begin{aligned}|\langle\nu_{L\downarrow}(\mathbf{p})|e^{-iHt}|\nu_{R\downarrow}(\mathbf{p})\rangle|^2 &= \sin^2(2\theta)\,\sin^2\Big(\frac{m_2^2 - m_1^2}{4E}L\Big)\\ &\approx 4\Big(\frac{m_D}{m_R}\Big)^2\sin^2\Big(\frac{Lm_R^2}{4E}\Big).\end{aligned} \tag{5.20}$$

The approximation follows from the seesaw assumption $m_R \gg m_D$, which was briefly mentioned at the end of chapter 5.2.

This result shows two characteristics that differ from the standard flavor-to-flavor oscillation.

First is the smallness of the oscillation amplitude, due to the factor $(m_D/m_R)^2$. This is a known feature of the canonical type I seesaw model [18]. The second is that if this oscillation can somehow be observed, the oscillation wavelength reveals information about the heavy lepton mass to a good approximation. However, the oscillation frequency depends on m_R^2/E, which implies that such a measurement is rather difficult if not impossible to perform.

6. Conclusions

In this work the conceptual framework presented in [21] was considered with two simple models of Majorana neutrinos, including the type I seesaw. To build the necessary foundations, the mathematical formalism of spin 1/2 particles and EW theory was studied, along with the standard treatment of the massive neutrino.

A one-to-one -correspondence between the SM flavor neutrino fields and the massive Majorana neutrino fields was established using Hamiltonian methods, which provides a framework for defining the oscillating neutrino states in the physical Fock space of the model, which is something that a pure Lagrangian formulation misses.

The results of the calculations carried out in this work agree with previously known results in the ultrarelativistic approximation, while extending them to the non-relativistic region similar to [21]. Furthermore, this work finds the same oscillation probability for the Majorana neutrinos as the probability for Dirac neutrinos found in [21]. This further backs up the ability of the method in formalizing the oscillating states in a proper quantum field theoretical setting, since a contradiction to existing knowledge has yet to be found.

However, many questions remain. To voice a few: how are oscillations in matter treated under this framework? Are there any insights to be gained to the production and absorption of massive neutrinos, i.e. to the interaction terms of the SM? Are there any complications if we extend from the simple models to, say, the three-flavor seesaw?

Indeed, future work among this framework is to be expected.

Appendix A. Spinor conventions

For all calculations in chapter 5, we will use the helicity eigenstates, introduced in chapter 2.5.1,

$$\chi^{\uparrow} = \chi^{R} = \begin{pmatrix} \cos\frac{\theta}{2}\, e^{-i\frac{\phi}{2}} \\ \sin\frac{\theta}{2}\, e^{i\frac{\phi}{2}} \end{pmatrix}, \tag{A.1}$$

$$\chi^{\downarrow} = \chi^{L} = \begin{pmatrix} \sin\frac{\theta}{2}\, e^{-i\frac{\phi}{2}} \\ -\cos\frac{\theta}{2}\, e^{i\frac{\phi}{2}} \end{pmatrix}, \tag{A.2}$$

as well as the following list of Dirac spinors in the Weyl basis:

$$u^{\uparrow}(\mathbf{p}) = \sqrt{\frac{E+m}{2}} \begin{pmatrix} (1-\frac{p}{E+m})\,\chi^{\uparrow} \\ (1+\frac{p}{E+m})\,\chi^{\uparrow} \end{pmatrix}, \qquad u^{\downarrow}(\mathbf{p}) = \sqrt{\frac{E+m}{2}} \begin{pmatrix} (1+\frac{p}{E+m})\,\chi^{\downarrow} \\ (1-\frac{p}{E+m})\,\chi^{\downarrow} \end{pmatrix}, \tag{A.3}$$

$$v^{\uparrow}(\mathbf{p}) = \sqrt{\frac{E+m}{2}} \begin{pmatrix} -(1+\frac{p}{E+m})\,\chi^{\downarrow} \\ (1-\frac{p}{E+m})\,\chi^{\downarrow} \end{pmatrix}, \qquad v^{\downarrow}(\mathbf{p}) = \sqrt{\frac{E+m}{2}} \begin{pmatrix} -(1-\frac{p}{E+m})\,\chi^{\uparrow} \\ (1+\frac{p}{E+m})\,\chi^{\uparrow} \end{pmatrix}, \tag{A.4}$$

where $p \equiv |\mathbf{p}|$ and $E = \sqrt{p^2+m^2}$.

From these, we retrieve the spinors with opposite 3-momentum, i.e. $u^{\uparrow}(-\mathbf{p})$. In spherical coordinates, the momentum inversion is achieved by taking $\theta \to \pi - \theta, \phi \to \pi + \phi$.

$$u^{\uparrow}(-\mathbf{p}) = i\sqrt{\frac{E+m}{2}} \begin{pmatrix} (1-\frac{p}{E+m})\,\chi^{\downarrow} \\ (1+\frac{p}{E+m})\,\chi^{\downarrow} \end{pmatrix}, \qquad u^{\downarrow}(-\mathbf{p}) = i\sqrt{\frac{E+m}{2}} \begin{pmatrix} (1+\frac{p}{E+m})\,\chi^{\uparrow} \\ (1-\frac{p}{E+m})\,\chi^{\uparrow} \end{pmatrix}, \tag{A.5}$$

$$v^{\uparrow}(-\mathbf{p}) = i\sqrt{\frac{E+m}{2}} \begin{pmatrix} -(1+\frac{p}{E+m})\,\chi^{\uparrow} \\ (1-\frac{p}{E+m})\,\chi^{\uparrow} \end{pmatrix}, \qquad v^{\downarrow}(-\mathbf{p}) = i\sqrt{\frac{E+m}{2}} \begin{pmatrix} -(1-\frac{p}{E+m})\,\chi^{\downarrow} \\ (1+\frac{p}{E+m})\,\chi^{\downarrow} \end{pmatrix}. \tag{A.6}$$

These results for opposite momentum spinors can be summarized with the substitution rule $\chi^{\lambda} \to i\chi^{-\lambda}$.

These spinors satisfy

$$-i\gamma^2[u^{\lambda}(\mathbf{p})]^* = v^{\lambda}(\mathbf{p}), \tag{A.7}$$

and the helicity eigenstates satisfy, by definition (2.46) as well as by straight calculation from (A.1, A.2), the properties

$$\sigma_i p^i \chi^\lambda = \mathrm{sgn}(\lambda)\, p \chi^\lambda, \tag{A.8}$$

$$\chi_\lambda^\dagger \chi_\lambda = 1, \tag{A.9}$$

$$\chi_\lambda^\dagger \chi_{(-\lambda)} = 0, \tag{A.10}$$

$$\chi_\lambda^T \sigma_2 \chi_\lambda = 0, \tag{A.11}$$

$$\chi_\lambda^T \sigma_2 \chi_{(-\lambda)} = i\ \mathrm{sgn}(\lambda) \tag{A.12}$$

where

$$\mathrm{sgn}(\lambda) = \begin{cases} +1, & \lambda = \uparrow \\ -1, & \lambda = \downarrow \end{cases} \tag{A.13}$$

Appendix B. Explicit calculations of chapter 5

For the sake of readability, we present here, instead of in the main text of chapter 5, in detail the original work that is behind the discussion of chapters 5.1 and 5.2.

B.1 Chapter 5.1

Recall the Lagrangian we start from:

$$\begin{aligned}\mathcal{L}_{\text{BSM}} &= i\sum_l \left\{\Psi^\dagger_{\nu_{l},L}\overline{\sigma}_\mu\partial^\mu\Psi_{\nu_{l},L}\right\} + \frac{i}{2}\sum_{ll'} m_{ll'}[\Psi^\dagger_{\nu_{l},L}\sigma_2\Psi^*_{\nu_{l'},L} - \Psi^T_{\nu_{l},L}\sigma_2\Psi_{\nu_{l'},L}]\\ &= \mathcal{L}_0 + \mathcal{L}_{\text{mass}}.\end{aligned} \tag{B.1}$$

Here $\overline{\sigma}_\mu$ is defined in (2.19), the fields are expressed in terms of their left-chiral components and we abbreviate $\Psi(\mathbf{x},t) = \Psi$. $\Psi_{\nu_{l},L}$ refers to the *mixed* neutrinos.

In Hamiltonian form (2.51) this reads

$$\begin{aligned}\mathcal{H}_{\text{BSM}} &= i\sum_l \left\{\Psi^\dagger_{\nu_{l},L}\sigma_i\partial^i\Psi_{\nu_{l},L}\right\} - \frac{i}{2}\sum_{ll'} m_{ll'}[\Psi^\dagger_{\nu_{l},L}\sigma_2\Psi^*_{\nu_{l'},L} - \Psi^T_{\nu_{l},L}\sigma_2\Psi_{\nu_{l'},L}]\\ &= \mathcal{H}_0 + \mathcal{H}_{\text{mass}}.\end{aligned} \tag{B.2}$$

Then we consider the Schrödinger picture (5.4) at $t = 0$, for this allows us to employ the field expansion (5.1), which becomes (we use the metric signature diag$(+,-,-,-)$)

$$\psi_{\nu_{l},L}(\mathbf{x},0) = \lim_{m\to 0} P_L\int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_\lambda\Big(a_{l\lambda}(\mathbf{p})u_\lambda(\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}} + b^\dagger_{l\lambda}(\mathbf{p})v_\lambda(\mathbf{p})e^{-i\mathbf{p}\cdot\mathbf{x}}\Big). \tag{B.3}$$

Treat the kinetic terms, $\mathcal{H}_0$, first.

Kinetic terms

The derivative of (B.3) yields

$$\begin{aligned}\sigma_i\partial^i\psi_{\nu_{l},L}(\mathbf{x},0) &= i\lim_{m\to 0} P_L\int\frac{d^3p}{(2\pi)^3}\frac{\sigma_i p^i}{\sqrt{2E_p}}\sum_\lambda\Big(a_{l\lambda}(\mathbf{p})u_\lambda(\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}} - b^\dagger_{l\lambda}(\mathbf{p})v_\lambda(\mathbf{p})e^{-i\mathbf{p}\cdot\mathbf{x}}\Big),\\ &= i\int_p\frac{\sigma_i p^i}{\sqrt{2p}}\sum_\lambda\Big(a_{l\lambda}(\mathbf{p})\Big[\lim_{m\to 0} P_L u_\lambda(\mathbf{p})\Big]e^{i\mathbf{p}\cdot\mathbf{x}} - b^\dagger_{l\lambda}(\mathbf{p})\Big[\lim_{m\to 0} P_L v_\lambda(\mathbf{p})\Big]e^{-i\mathbf{p}\cdot\mathbf{x}}\Big),\end{aligned} \tag{B.4}$$

and the conjugate transpose of (B.3) is

$$\psi^\dagger_{\nu_l,L}(\mathbf{x},0) = \int_p \frac{1}{\sqrt{2p}} \sum_\lambda \left(a^\dagger_{l\lambda}(\mathbf{p}) \Big[\lim_{m\to 0} P_L u_\lambda(\mathbf{p}) \Big]^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}} + b_{l\lambda}(\mathbf{p}) \Big[\lim_{m\to 0} P_L v_\lambda(\mathbf{p}) \Big]^\dagger e^{i\mathbf{p}\cdot\mathbf{x}} \right), \tag{B.5}$$

where we introduced the shorthand $\int_p = \int d^3p/(2\pi)^3$. It is tempting to evaluate the limit and the projection at this point, but we'd lose out on important details if we didn't wait until we've evaluated some δ -functions.

Let $\int_x = \int d^3x$, and insert the above expansions to the kinetic Hamiltonian:

$$\begin{aligned} H_0 = \int_x \mathcal{H}_0 = -\sum_{l,\lambda,\lambda'} \int_{p,q,x} \frac{1}{2\sqrt{pq}} \Big\{ & a^\dagger_{l\lambda}(\mathbf{p}) a_{l\lambda'}(\mathbf{q}) e^{i(\mathbf{q}-\mathbf{p})\cdot\mathbf{x}} \Big[\lim_{m\to 0} P_L u_\lambda(\mathbf{p}) \Big]^\dagger \sigma_i q^i \Big[\lim_{m\to 0} P_L u_{\lambda'}(\mathbf{q}) \Big] \\ & - a^\dagger_{l\lambda}(\mathbf{p}) b^\dagger_{l\lambda'}(\mathbf{q}) e^{-i(\mathbf{q}+\mathbf{p})\cdot\mathbf{x}} \Big[\lim_{m\to 0} P_L u_\lambda(\mathbf{p}) \Big]^\dagger \sigma_i q^i \Big[\lim_{m\to 0} P_L v_{\lambda'}(\mathbf{q}) \Big] \\ & + b_{l\lambda}(\mathbf{p}) a_{l\lambda'}(\mathbf{q}) e^{i(\mathbf{q}+\mathbf{p})\cdot\mathbf{x}} \Big[\lim_{m\to 0} P_L v_\lambda(\mathbf{p}) \Big]^\dagger \sigma_i q^i \Big[\lim_{m\to 0} P_L u_{\lambda'}(\mathbf{q}) \Big] \\ & - b_{l\lambda}(\mathbf{p}) b^\dagger_{l\lambda'}(\mathbf{q}) e^{i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}} \Big[\lim_{m\to 0} P_L v_\lambda(\mathbf{p}) \Big]^\dagger \sigma_i q^i \Big[\lim_{m\to 0} P_L v_{\lambda'}(\mathbf{q}) \Big] \Big\}. \end{aligned} \tag{B.6}$$

To proceed from here, recall the definition of the Dirac δ -function,

$$(2\pi)^3 \delta^3(\mathbf{p}-\mathbf{q}) = \int_x e^{\pm i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}}, \tag{B.7}$$

which we use to evaluate the integrals over x and q.

$$\begin{aligned} H_0 = -\sum_{l,\lambda,\lambda'} \int_p \frac{1}{2p} \Big\{ & a^\dagger_{l\lambda}(\mathbf{p}) a_{l\lambda'}(\mathbf{p}) \Big[\lim_{m\to 0} P_L u_\lambda(\mathbf{p}) \Big]^\dagger \sigma_i p^i \Big[\lim_{m\to 0} P_L u_{\lambda'}(\mathbf{p}) \Big] \\ & - a^\dagger_{l\lambda}(\mathbf{p}) b^\dagger_{l\lambda'}(-\mathbf{p}) \Big[\lim_{m\to 0} P_L u_\lambda(\mathbf{p}) \Big]^\dagger \sigma_i (-p)^i \Big[\lim_{m\to 0} P_L v_{\lambda'}(-\mathbf{p}) \Big] \\ & + b_{l\lambda}(\mathbf{p}) a_{l\lambda'}(-\mathbf{p}) \Big[\lim_{m\to 0} P_L v_\lambda(\mathbf{p}) \Big]^\dagger \sigma_i (-p)^i \Big[\lim_{m\to 0} P_L u_{\lambda'}(-\mathbf{p}) \Big] \\ & - b_{l\lambda}(\mathbf{p}) b^\dagger_{l\lambda'}(\mathbf{p}) \Big[\lim_{m\to 0} P_L v_\lambda(\mathbf{p}) \Big]^\dagger \sigma_i p^i \Big[\lim_{m\to 0} P_L v_{\lambda'}(\mathbf{p}) \Big] \Big\}. \end{aligned} \tag{B.8}$$

Now we may take the limits and projections of the spinors (A.3 - A.6).

$$\lim_{m\to 0} P_L u_\lambda(\mathbf{p}) = \begin{cases} 0, & \lambda = \uparrow \\ \sqrt{2p}\,\chi_\downarrow, & \lambda = \downarrow \end{cases} \tag{B.9}$$

$$\lim_{m\to 0} P_L v_\lambda(\mathbf{p}) = \begin{cases} -\sqrt{2p}\,\chi_\downarrow, & \lambda = \uparrow \\ 0, & \lambda = \downarrow \end{cases} \tag{B.10}$$

$$\lim_{m\to 0} P_L u_\lambda(-\mathbf{p}) = \begin{cases} 0, & \lambda = \uparrow \\ i\sqrt{2p}\chi_\uparrow, & \lambda = \downarrow \end{cases} \tag{B.11}$$

$$\lim_{m\to 0} P_L v_\lambda(-\mathbf{p}) = \begin{cases} -i\sqrt{2p}\,\chi_\uparrow, & \lambda = \uparrow \\ 0, & \lambda = \downarrow \end{cases}, \tag{B.12}$$

where $p = \sqrt{\mathbf{p}^2}$. This reduces the Hamiltonian into

$$
\begin{aligned}
H_0 = -\sum_l \int_p \Big\{ & a^\dagger_{l\downarrow}(\mathbf{p}) a_{l\downarrow}(\mathbf{p}) [\chi^\dagger_\downarrow \sigma_i p^i \chi_\downarrow] \\
& - a^\dagger_{l\downarrow}(\mathbf{p}) b^\dagger_{l\uparrow}(-\mathbf{p}) [-i\chi^\dagger_\downarrow \sigma_i (-p)^i \chi_\uparrow] \\
& + b_{l\uparrow}(\mathbf{p}) a_{l\downarrow}(-\mathbf{p}) [-i\chi^\dagger_\downarrow \sigma_i (-p)^i \chi_\uparrow] \\
& - b_{l\uparrow}(\mathbf{p}) b^\dagger_{l\uparrow}(\mathbf{p}) [\chi^\dagger_\downarrow \sigma_i p^i \chi_\downarrow] \Big\},
\end{aligned} \tag{B.13}
$$

in which the quantities in brackets $[\cdot]$ are evaluated using the properties of the helicity eigenstates (A.8 - A.10): $[\chi^\dagger_\downarrow \sigma_i p^i \chi_\downarrow] = -p$, $[-i\chi^\dagger_\downarrow \sigma_i (-p)^i \chi_\uparrow] = 0$. Hence,

$$
\begin{aligned}
H_0 &= -\sum_l \int_p -p \Big\{ a^\dagger_{l\downarrow}(\mathbf{p}) a_{l\downarrow}(\mathbf{p}) - b_{l\uparrow}(\mathbf{p}) b^\dagger_{l\uparrow}(\mathbf{p}) \Big\} \\
&= \sum_l \int_p p \Big\{ a^\dagger_{l\downarrow}(\mathbf{p}) a_{l\downarrow}(\mathbf{p}) + b^\dagger_{l\uparrow}(\mathbf{p}) b_{l\uparrow}(\mathbf{p}) \Big\},
\end{aligned} \tag{B.14}
$$

in which the last equality is the standard normal-ordering process, where an infinity has been discarded.

Mass terms

Then, consider the mass terms. Let's introduce an abuse of notation, by writing

$$
\mathcal{H}_{\text{mass}} = \frac{i}{2} \sum_{ll'} m_{ll'} \Psi^T_{\nu_{l,L}} \sigma_2 \Psi_{\nu_{l',L}} - c.c. \tag{B.15}
$$

where *c.c.* denotes the complex conjugate of complex numbers. To reiterate: when writing this, we are **not** using the usual definition of complex conjugation for Grassmann numbers which satisfies $(\overline{\psi}\chi)^* = \overline{\chi}\psi$. Instead, with this notation, we mean $[i\Psi^\dagger_{\nu_{l,L}} \sigma_2 \Psi^*_{\nu_{l',L}}]^* = -i[\Psi^\dagger_{\nu_{l,L}}]^* \sigma_2^* [\Psi^*_{\nu_{l',L}}]^* = i\Psi^T_{\nu_{l,L}} \sigma_2 \Psi_{\nu_{l',L}}$. The reasoning for this notation is that we wish to retain the order of the flavors l and l' in the expression.

The transpose of (B.3) yields

$$
\psi^T_{\nu_{l,L}}(\mathbf{x}, 0) = \int_p \frac{1}{\sqrt{2p}} \sum_\lambda \Big(a_{l\lambda}(\mathbf{p}) \Big[\lim_{m\to 0} P_L u_\lambda(\mathbf{p}) \Big]^T e^{i\mathbf{p}\cdot\mathbf{x}} + b^\dagger_{l\lambda}(\mathbf{p}) \Big[\lim_{m\to 0} P_L v_\lambda(\mathbf{p}) \Big]^T e^{-i\mathbf{p}\cdot\mathbf{x}} \Big). \tag{B.16}
$$

The Hamiltonian becomes

$$
\begin{aligned}
H_{\text{mass}} = \frac{i}{2} \sum_{l,l',\lambda,\lambda'} m_{ll'} \int_{x,p,q} \frac{1}{2\sqrt{pq}} \Big\{ & a_{l\lambda}(\mathbf{p}) a_{l'\lambda'}(\mathbf{q}) e^{i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}} \Big[\lim_{m\to 0} P_L u_\lambda(\mathbf{p}) \Big]^T \sigma_2 \Big[\lim_{m\to 0} P_L u_{\lambda'}(\mathbf{q}) \Big] \\
& + a_{l\lambda}(\mathbf{p}) b^\dagger_{l'\lambda'}(\mathbf{q}) e^{i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}} \Big[\lim_{m\to 0} P_L u_\lambda(\mathbf{p}) \Big]^T \sigma_2 \Big[\lim_{m\to 0} P_L v_{\lambda'}(\mathbf{q}) \Big] \\
& + b^\dagger_{l\lambda}(\mathbf{p}) a_{l'\lambda'}(\mathbf{q}) e^{-i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}} \Big[\lim_{m\to 0} P_L v_\lambda(\mathbf{p}) \Big]^T \sigma_2 \Big[\lim_{m\to 0} P_L u_{\lambda'}(\mathbf{q}) \Big] \\
& + b^\dagger_{l\lambda}(\mathbf{p}) b^\dagger_{l'\lambda'}(\mathbf{q}) e^{-i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}} \Big[\lim_{m\to 0} P_L v_\lambda(\mathbf{p}) \Big]^T \sigma_2 \Big[\lim_{m\to 0} P_L v_{\lambda'}(\mathbf{q}) \Big] \Big\} - c.c.,
\end{aligned} \tag{B.17}
$$

into which we again insert the δ -functions to find

$$\begin{aligned}H_{\text{mass}} = \frac{i}{2}\sum_{l,l',\lambda,\lambda'} m_{ll'}\int_p \frac{1}{2p}\Big\{ & a_{l\lambda}(\mathbf{p})a_{l'\lambda'}(-\mathbf{p})\Big[\lim_{m\to 0} P_L u_\lambda(\mathbf{p})\Big]^T\sigma_2\Big[\lim_{m\to 0} P_L u_{\lambda'}(-\mathbf{p})\Big] \\ & + a_{l\lambda}(\mathbf{p})b^\dagger_{l'\lambda'}(\mathbf{p})\Big[\lim_{m\to 0} P_L u_\lambda(\mathbf{p})\Big]^T\sigma_2\Big[\lim_{m\to 0} P_L v_{\lambda'}(\mathbf{p})\Big] \\ & + b^\dagger_{l\lambda}(\mathbf{p})a_{l'\lambda'}(\mathbf{p})\Big[\lim_{m\to 0} P_L v_\lambda(\mathbf{p})\Big]^T\sigma_2\Big[\lim_{m\to 0} P_L u_{\lambda'}(\mathbf{p})\Big] \\ & + b^\dagger_{l\lambda}(\mathbf{p})b^\dagger_{l'\lambda'}(-\mathbf{p})\Big[\lim_{m\to 0} P_L v_\lambda(\mathbf{p})\Big]^T\sigma_2\Big[\lim_{m\to 0} P_L v_{\lambda'}(-\mathbf{p})\Big]\Big\} - c.c. \end{aligned} \tag{B.18}$$

Evaluation of the limit and the projection gives

$$\begin{aligned}H_{\text{mass}} = \frac{i}{2}\sum_{l,l'} m_{ll'}\int_p \Big\{ & a_{l\downarrow}(\mathbf{p})a_{l'\downarrow}(-\mathbf{p})[i\chi_\downarrow^T\sigma_2\chi_\uparrow] \\ & - a_{l\downarrow}(\mathbf{p})b^\dagger_{l'\uparrow}(\mathbf{p})[\chi_\downarrow^T\sigma_2\chi_\downarrow] \\ & - b^\dagger_{l\uparrow}(\mathbf{p})a_{l'\downarrow}(\mathbf{p})[\chi_\downarrow^T\sigma_2\chi_\downarrow] \\ & + b^\dagger_{l\uparrow}(\mathbf{p})b^\dagger_{l'\uparrow}(-\mathbf{p})[i\chi_\downarrow^T\sigma_2\chi_\uparrow]\Big\} - c.c. \end{aligned} \tag{B.19}$$

Using the properties of the helicity eigenstates (A.11, A.12), we retrieve the identities $[i\chi_\downarrow^T\sigma_2\chi_\uparrow] = 1, [\chi_\downarrow^T\sigma_2\chi_\downarrow] = 0$, which we insert to the Hamiltonian.

$$H_{\text{mass}} = \frac{i}{2}\sum_{l,l'} m_{ll'}\int_p \Big\{a_{l\downarrow}(\mathbf{p})a_{l'\downarrow}(-\mathbf{p}) + b^\dagger_{l\uparrow}(\mathbf{p})b^\dagger_{l'\uparrow}(-\mathbf{p})\Big\} - c.c. \tag{B.20}$$

Diagonalizing the flavors

To this form, we apply a simple rotation to diagonalize it in terms of flavors. For the 2-flavor case an appropriate rotation is

$$\begin{pmatrix} a_{e\lambda}(\mathbf{p}) \\ a_{\mu\lambda}(\mathbf{p})\end{pmatrix} = \begin{pmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix} a_{1\lambda}(\mathbf{p}) \\ a_{2\lambda}(\mathbf{p})\end{pmatrix}, \qquad \tan 2\theta = \frac{2m_{e\mu}}{m_{\mu\mu} - m_{ee}}, \tag{B.21}$$

and an equivalent rotation for operators b.* This rotation is unitary, hence the kinetic terms and the CCR, are unaffected. Note that rotating the quantum operators in this manner is equivalent to rotating the fields

$$\begin{pmatrix}\Psi_{\nu_e} \\ \Psi_{\nu_\mu}\end{pmatrix} = \begin{pmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\Psi_1 \\ \Psi_2\end{pmatrix}. \tag{B.22}$$

*Do note that for simplicity we consider a real unitary matrix, not a complex unitary matrix as we should. A complex matrix is not really relevant for the conclusions of this calculation, but results in a need to introduce absolute values in the seesaw case. As a reminder that the masses are indeed positive if the rotation is done properly.

Applying this rotation, we find

$$H_{\text{mass}} = \frac{i}{2}\sum_i m_i \int_p \Big\{ a_{i\downarrow}(\mathbf{p}) a_{i\downarrow}(-\mathbf{p}) + b^\dagger_{i\uparrow}(\mathbf{p}) b^\dagger_{i\uparrow}(-\mathbf{p}) \Big\} - c.c., \tag{B.23}$$

where the masses are linear combinations of the parameters $m_{ll'}$ that we began with,

$$\begin{aligned} m_1 &= m_{ee}\cos^2\theta + m_{\mu\mu}\sin^2\theta - 2m_{e\mu}\cos\theta\,\sin\theta, \\ m_2 &= m_{ee}\sin^2\theta + m_{\mu\mu}\cos^2\theta + 2m_{e\mu}\cos\theta\,\sin\theta. \end{aligned} \tag{B.24}$$

Finishing the normal-ordering

The result we've thus far obtained, including the complex conjugate terms, is

$$\begin{aligned} H = H_0 + H_{\text{mass}} = \sum_i \int_p p \Big\{ & a^\dagger_{i\downarrow}(\mathbf{p}) a_{i\downarrow}(\mathbf{p}) + b^\dagger_{i\uparrow}(\mathbf{p}) b_{i\uparrow}(\mathbf{p}) \Big\} \\ & + \frac{i}{2} m_i \Big\{ a_{i\downarrow}(\mathbf{p}) a_{i\downarrow}(-\mathbf{p}) + b^\dagger_{i\uparrow}(\mathbf{p}) b^\dagger_{i\uparrow}(-\mathbf{p}) \\ & \qquad + a^\dagger_{i\downarrow}(\mathbf{p}) a^\dagger_{i\downarrow}(-\mathbf{p}) + b_{i\uparrow}(\mathbf{p}) b_{i\uparrow}(-\mathbf{p}) \Big\}. \end{aligned} \tag{B.25}$$

Noteworthy is that the antiparticle operator, b, associated with the SM flavor neutrino is seen only with helicity up (right), and the particle operator a only with helicity down (left). The origin of this is the massless L-chiral projection. This allows us to write

$$\begin{aligned} c_{i\downarrow} &= a_{i\downarrow}, \\ c_{i\uparrow} &= b_{i\uparrow}, \end{aligned} \tag{B.26}$$

which we use to compactify the notation:

$$H = \sum_{i,\lambda} \int_p p\, c^\dagger_{i\lambda}(\mathbf{p}) c_{i\lambda}(\mathbf{p}) + \frac{i}{2} m_i \Big\{ c^\dagger_{i\lambda}(\mathbf{p}) c^\dagger_{i\lambda}(-\mathbf{p}) + c_{i\lambda}(\mathbf{p}) c_{i\lambda}(-\mathbf{p}) \Big\}. \tag{B.27}$$

To this form we introduce the Bogoliubov transformation [67]

$$A_{i\lambda}(\mathbf{p}) = \alpha_{ip} c_{i\lambda}(\mathbf{p}) + \beta_{ip} c^\dagger_{i\lambda}(-\mathbf{p}), \tag{B.28}$$

where the constants are found to be

$$\alpha_{ip} = \sqrt{\frac{1}{2}\Big(1 + \frac{p}{\Omega_{ip}}\Big)}, \quad \beta_{ip} = i\sqrt{\frac{1}{2}\Big(1 - \frac{p}{\Omega_{ip}}\Big)}, \quad \Omega_{ip} = \sqrt{p^2 + m_i^2}. \tag{B.29}$$

Note that these operators $A_{i\lambda}(\mathbf{p})$ satisfy the same CCR as the operators $c_{i\lambda}(\mathbf{p})$:

$$\begin{aligned} \{A_{i\lambda}(\mathbf{p}), A^\dagger_{j\lambda'}(\mathbf{q})\} &= \alpha_{ip}\alpha^\dagger_{jq}\{c_{i\lambda}(\mathbf{p}), c^\dagger_{j\lambda'}(\mathbf{q})\} + \beta_{ip}\beta^\dagger_{jq}\{c^\dagger_{i\lambda}(-\mathbf{p}), c_{j\lambda'}(-\mathbf{q})\} \\ &= (|\alpha_{ip}|^2 - |\beta_{ip}|^2)\delta_{ij}\delta_{\lambda\lambda'}\delta^3(\mathbf{p} - \mathbf{q}) \\ &= \delta_{ij}\delta_{\lambda\lambda'}\delta^3(\mathbf{p} - \mathbf{q}). \end{aligned} \tag{B.30}$$

After inserting the transformation, we find the Hamiltonian as

$$H = \sum_{i,\lambda} \int_p \Omega_{ip} A^\dagger_{i\lambda}(\mathbf{p}) A_{i\lambda}(\mathbf{p}), \tag{B.31}$$

which corresponds to two free and massive Majorana fields, for there is only one kind of an operator present for given helicity and energy Ω_{ip}.

The vacuum state

We proceed to confirm that the state

$$|\Phi_0\rangle = \prod_{i,\lambda,p} \Big(\alpha_{ip} - \beta_{ip} c^\dagger_{i\lambda}(\mathbf{p}) c^\dagger_{i\lambda}(-\mathbf{p}) \Big) |0\rangle \tag{B.32}$$

is indeed the state which is annihilated by the operator $A_{i\lambda}(\mathbf{p})$, thereby making it the vacuum state of the theory. Recall (2.55 - 2.57), (2.60) and (2.65), which give

$$\begin{aligned} \alpha_{jq} c_{j\lambda'}(\mathbf{q}) |\Phi_0\rangle &= \alpha_{jq} c_{j\lambda'}(\mathbf{q}) \prod_{i,\lambda,p} \Big(\alpha_{ip} - \beta_{ip} c^\dagger_{i\lambda}(\mathbf{p}) c^\dagger_{i\lambda}(-\mathbf{p}) \Big) |0\rangle \\ &\propto -\alpha_{jq}\beta_{jq} c_{j\lambda'}(\mathbf{q}) c^\dagger_{j\lambda'}(\mathbf{q}) c^\dagger_{j\lambda'}(-\mathbf{q}) |0\rangle \\ &= -\alpha_{jq}\beta_{jq} c^\dagger_{j\lambda'}(-\mathbf{q}) |0\rangle, \end{aligned} \tag{B.33}$$

and

$$\begin{aligned} \beta_{jq} c^\dagger_{j\lambda'}(\mathbf{q}) |\Phi_0\rangle &= \beta_{jq} c^\dagger_{j\lambda'}(-\mathbf{q}) \prod_{i,\lambda,p} \Big(\alpha_{ip} - \beta_{ip} c^\dagger_{i\lambda}(\mathbf{p}) c^\dagger_{i\lambda}(-\mathbf{p}) \Big) |0\rangle \\ &\propto \alpha_{jq}\beta_{jq} c^\dagger_{j\lambda'}(-\mathbf{q}) |0\rangle - \beta^2_{jq} c^\dagger_{j\lambda'}(\mathbf{q}) c^\dagger_{j\lambda'}(\mathbf{q}) c^\dagger_{j\lambda'}(-\mathbf{q}) |0\rangle \\ &= \alpha_{jq}\beta_{jq} c^\dagger_{j\lambda'}(-\mathbf{q}) |0\rangle, \end{aligned} \tag{B.34}$$

which confirm the claimed property.

We remind that the operators $c^\dagger_{i\lambda}(\mathbf{p})$ and $c^\dagger_{i\lambda}(-\mathbf{p})$ have opposite spins, as λ refers to helicity.

Vacuum overlap

Finally, we confirm that the vacua have a vanishing overlap, which can be seen by inspecting the infinite momentum limit:

$$
\begin{aligned}
\langle 0|\Phi_0\rangle &= \prod_{p\neq 0}\frac{1}{2}\sqrt{\Big(1+\frac{p}{\Omega_{1p}}\Big)\Big(1+\frac{p}{\Omega_{2p}}\Big)} \\
&\overset{p\gg m_i}{\propto} \prod_{p}\frac{1}{2}\sqrt{\Big(2-\frac{m_1^2}{2p^2}\Big)}\sqrt{\Big(2-\frac{m_2^2}{2p^2}\Big)} \\
&= \exp\Big\{\frac{V}{2}\int_p \ln\sqrt{\Big(1-\frac{m_1^2}{4p^2}\Big)\Big(1-\frac{m_2^2}{4p^2}\Big)}\Big\} \\
&\approx \exp\Big\{\frac{-V(m_1^2+m_2^2)}{4}\int_p\frac{1}{4p^2}\Big\} \\
&= \exp\Big\{\frac{-V(m_1^2+m_2^2)}{2(4\pi)^2}\int_\Lambda^{\Lambda'} dp\Big\},
\end{aligned}
\tag{B.35}
$$

where Λ indicates approximately the scale where $p \gg m_i$; $\Lambda' \to \infty$; and we used the series expansion of $\ln(1-x)$ at small x. The volume element V appears when we go from discrete momenta to a continuum, $\sum_p \to \frac{V}{2}\int_p$ [60, Chapter 10.5].

Hence the inner product $\langle 0|\Phi_0\rangle \to 0$ as $\Lambda' \to \infty$, as well as when $V \to \infty$.

Inverse Bogoliubov transform

For future purposes, we remark that the inverted Bogoliubov transformation satisfies

$$
c_{i\lambda}(\mathbf{p}) = \alpha_{ip}A_{i\lambda}(\mathbf{p}) - \beta_{ip}A^\dagger_{i\lambda}(-\mathbf{p}),
\tag{B.36}
$$

in which

$$
A_{i,\lambda}(-\mathbf{p}) = \alpha_{ip}c_{i\lambda}(-\mathbf{p}) - \beta_{ip}c^\dagger_{i\lambda}(\mathbf{p}).
\tag{B.37}
$$

This leads to the flavor neutrino operators

$$
\begin{aligned}
a_{e\downarrow}(\mathbf{p}) &= \cos(\theta)\Big(\alpha_{1p}A_{1\downarrow}(\mathbf{p}) - \beta_{1p}A^\dagger_{1\downarrow}(-\mathbf{p})\Big) + \sin(\theta)\Big(\alpha_{2p}A_{2\downarrow}(\mathbf{p}) - \beta_{2p}A^\dagger_{2\downarrow}(-\mathbf{p})\Big),\\
a_{\mu\downarrow}(\mathbf{p}) &= \cos(\theta)\Big(\alpha_{2p}A_{2\downarrow}(\mathbf{p}) - \beta_{2p}A^\dagger_{2\downarrow}(-\mathbf{p})\Big) - \sin(\theta)\Big(\alpha_{1p}A_{1\downarrow}(\mathbf{p}) - \beta_{1p}A^\dagger_{1\downarrow}(-\mathbf{p})\Big),
\end{aligned}
\tag{B.38}
$$

and equivalent relations for $b_{l\uparrow}(\mathbf{p})$.

B.2 Chapter 5.2

The calculation follows a process identical to the one presented above, with some differences that come with the L-R -crossterms and the R-chiral terms. We find the

Hamiltonian corresponding to (5.10) to be

$$\begin{aligned}\mathcal{H}_{\mathrm{BSM}} = & \, i\Psi^{\dagger}_{\nu,L}\sigma_i\partial^i\Psi_{\nu,L} + i\Psi^{\dagger}_{\nu,R}\overline{\sigma}_i\partial^i\Psi_{\nu,R} \\ & - \frac{1}{2}m_D[\Psi^{\dagger}_L\Psi_R - \Psi^T_L\Psi^*_R + h.c.] \\ & - \frac{i}{2}m_R[\Psi^T_R\sigma_2\Psi_R - \Psi^{\dagger}_R\sigma_2\Psi^*_R],\end{aligned} \tag{B.39}$$

where h.c. denotes the hermitian conjugate of the second row, which satisfies $\Psi^{\dagger}_L\Psi_R + h.c. = \Psi^{\dagger}_L\Psi_R + \Psi^{\dagger}_R\Psi_L$. In other words, for the $+h.c.$ notation we use the proper conjugation of Grassmann variables, in contrast with the $c.c.$ notation used in a previous section.

We identify the Schrödinger picture at $t = 0$ with (5.11). The mode expansion for $\psi_{\nu,L}(\mathbf{x}, 0)$ satisfies a form similar to (B.3) and

$$\psi_{\nu,R}(\mathbf{x}, 0) = \lim_{m\to 0} P_R \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_{\lambda}\Big(a_{R\lambda}(\mathbf{p})u_{\lambda}(\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}} + b^{\dagger}_{R\lambda}(\mathbf{p})v_{\lambda}(\mathbf{p})e^{-i\mathbf{p}\cdot\mathbf{x}}\Big), \tag{B.40}$$

where the operators a_R, b_R are the particle- and antiparticle operators corresponding to the massless R-chiral lepton and P_R is the right-chiral projection (2.38).

Kinetic terms

The R-chiral kinetic term yields

$$\begin{aligned}H_{0,R} = \sum_{\lambda,\lambda'}\int_p \frac{1}{2p}\Big\{ & a^{\dagger}_{R\lambda}(\mathbf{p})a_{R\lambda'}(\mathbf{p})\Big[\lim_{m\to 0}P_R u_{\lambda}(\mathbf{p})\Big]^{\dagger}\sigma_i p^i\Big[\lim_{m\to 0}P_R u_{\lambda'}(\mathbf{p})\Big] \\ & - a^{\dagger}_{R\lambda}(\mathbf{p})b^{\dagger}_{R\lambda'}(-\mathbf{p})\Big[\lim_{m\to 0}P_R u_{\lambda}(\mathbf{p})\Big]^{\dagger}\sigma_i(-p^i)\Big[\lim_{m\to 0}P_R v_{\lambda'}(-\mathbf{p})\Big] \\ & + b_{R\lambda}(\mathbf{p})a_{R\lambda'}(-\mathbf{p})\Big[\lim_{m\to 0}P_R v_{\lambda}(\mathbf{p})\Big]^{\dagger}\sigma_i(-p^i)\Big[\lim_{m\to 0}P_R u_{\lambda'}(-\mathbf{p})\Big] \\ & - b_{R\lambda}(\mathbf{p})b^{\dagger}_{R\lambda'}(\mathbf{p})\Big[\lim_{m\to 0}P_R v_{\lambda}(\mathbf{p})\Big]^{\dagger}\sigma_i p^i\Big[\lim_{m\to 0}P_R v_{\lambda'}(\mathbf{p})\Big]\Big\}.\end{aligned} \tag{B.41}$$

We insert the R-chiral projections

$$\lim_{m\to 0} P_R u_\lambda(\mathbf{p}) = \begin{cases} \sqrt{2p}\,\chi_\uparrow, & \lambda = \uparrow \\ 0, & \lambda = \downarrow \end{cases} \tag{B.42}$$

$$\lim_{m\to 0} P_R v_\lambda(\mathbf{p}) = \begin{cases} 0, & \lambda = \uparrow \\ \sqrt{2p}\,\chi_\uparrow, & \lambda = \downarrow \end{cases} \tag{B.43}$$

$$\lim_{m\to 0} P_R u_\lambda(-\mathbf{p}) = \begin{cases} i\sqrt{2p}\chi_\downarrow, & \lambda = \uparrow \\ 0, & \lambda = \downarrow \end{cases} \tag{B.44}$$

$$\lim_{m\to 0} P_R v_\lambda(-\mathbf{p}) = \begin{cases} 0, & \lambda = \uparrow \\ i\sqrt{2p}\chi_\downarrow, & \lambda = \downarrow \end{cases} \tag{B.45}$$

the spinor identities $[\chi_\uparrow^\dagger \sigma_i p^i \chi_\uparrow] = p$, $[\chi_\uparrow^\dagger i\sigma_i(-p^i)\chi_\downarrow] = 0$, and apply normal ordering to find

$$H_{0,R} = \int_p p\Big\{a_{R\uparrow}^\dagger(\mathbf{p})a_{R\uparrow}(\mathbf{p}) + b_{R\downarrow}^\dagger(\mathbf{p})b_{R\downarrow}(\mathbf{p})\Big\}, \tag{B.46}$$

which is almost exactly as the L-chiral case for one flavor (B.14), only with different operators, and in the R-chiral case the operator associated with the antiparticle spinor, f, has helicity down and d has helicity up, which mirrors the L-chiral case.

Hence, the kinetic terms combined equal

$$H_0 = H_{0,R} + H_{0,L} = \int_p p\Big\{a_{R\uparrow}^\dagger(\mathbf{p})a_{R\uparrow}(\mathbf{p}) + b_{R\downarrow}^\dagger(\mathbf{p})b_{R\downarrow}(\mathbf{p}) + a_{L\downarrow}^\dagger(\mathbf{p})a_{L\downarrow}(\mathbf{p}) + b_{L\uparrow}^\dagger(\mathbf{p})b_{L\uparrow}(\mathbf{p})\Big\}. \tag{B.47}$$

Mass terms

First we treat the terms with m_R. Note that with the $h.c.$ -notation we have

$$\mathcal{H}_{m_R} = -\frac{i}{2}m_R \Psi_R^T \sigma_2 \Psi_R + h.c. \tag{B.48}$$

This yields

$$\begin{aligned} H_{m_R} = -\frac{i}{2}m_R \sum_{\lambda,\lambda'} \int_p \frac{1}{2p}\Big\{ & a_{R\lambda}(\mathbf{p})a_{R\lambda'}(-\mathbf{p})\Big[\lim_{m\to 0} P_R u_\lambda(\mathbf{p})\Big]^T \sigma_2 \Big[\lim_{m\to 0} P_R u_{\lambda'}(-\mathbf{p})\Big] \\ & + a_{R\lambda}(\mathbf{p})b_{R\lambda'}^\dagger(\mathbf{p})\Big[\lim_{m\to 0} P_R u_\lambda(\mathbf{p})\Big]^T \sigma_2 \Big[\lim_{m\to 0} P_R v_{\lambda'}(\mathbf{p})\Big] \\ & + b_{R\lambda}^\dagger(\mathbf{p})a_{R\lambda'}(\mathbf{p})\Big[\lim_{m\to 0} P_R v_\lambda(\mathbf{p})\Big]^T \sigma_2 \Big[\lim_{m\to 0} P_R u_{\lambda'}(\mathbf{p})\Big] \\ & + b_{R\lambda}^\dagger(\mathbf{p})b_{R\lambda'}^\dagger(-\mathbf{p})\Big[\lim_{m\to 0} P_R v_\lambda(\mathbf{p})\Big]^T \sigma_2 \Big[\lim_{m\to 0} P_R v_{\lambda'}(-\mathbf{p})\Big]\Big\} + h.c., \end{aligned} \tag{B.49}$$

which evaluates to

$$H_{m_R} = \frac{i}{2} m_R \int_p \Big\{ a_{R\uparrow}(\mathbf{p}) a_{R\uparrow}(-\mathbf{p}) + b^\dagger_{R\downarrow}(\mathbf{p}) b^\dagger_{R\downarrow}(-\mathbf{p}) \Big\} + h.c., \tag{B.50}$$

where we used the spinor identities $[\chi_\uparrow^T \sigma_2 \chi_\downarrow] = i$, $[\chi_\uparrow^T \sigma_2 \chi_\uparrow] = 0$. The result again mirrors the L-chiral case, as expected.

Next, consider the terms with m_D. We evaluate the products

$$\begin{aligned}
\int_x \Psi_L^T \Psi_R^* = \sum_{\lambda,\lambda'} \int_p \frac{1}{2p} \Big\{ & a_{L\lambda}(\mathbf{p}) a^\dagger_{R\lambda'}(\mathbf{p}) \Big[\lim_{m\to 0} P_L u_\lambda(\mathbf{p}) \Big]^T \Big[\lim_{m\to 0} P_R u_{\lambda'}(\mathbf{p}) \Big]^* \\
& + a_{L\lambda}(\mathbf{p}) b_{R\lambda'}(-\mathbf{p}) \Big[\lim_{m\to 0} P_L u_\lambda(\mathbf{p}) \Big]^T \Big[\lim_{m\to 0} P_R v_{\lambda'}(-\mathbf{p}) \Big]^* \\
& + b^\dagger_{L\lambda}(\mathbf{p}) a^\dagger_{R\lambda'}(-\mathbf{p}) \Big[\lim_{m\to 0} P_L v_\lambda(\mathbf{p}) \Big]^T \Big[\lim_{m\to 0} P_R u_{\lambda'}(-\mathbf{p}) \Big]^* \\
& + b^\dagger_{L\lambda}(\mathbf{p}) b_{R\lambda'}(\mathbf{p}) \Big[\lim_{m\to 0} P_L v_\lambda(\mathbf{p}) \Big]^T \Big[\lim_{m\to 0} P_R v_{\lambda'}(\mathbf{p}) \Big]^* \Big\} \\
= \sum_{\lambda,\lambda'} \int_p \Big\{ & a_{L\lambda}(\mathbf{p}) a^\dagger_{R\lambda'}(\mathbf{p}) [\delta_{\lambda\downarrow} \delta_{\lambda'\uparrow} \chi_\downarrow^T \chi_\uparrow^*] \\
& + a_{L\lambda}(\mathbf{p}) b_{R\lambda'}(-\mathbf{p}) [-i \delta_{\lambda\downarrow} \delta_{\lambda'\downarrow} \chi_\downarrow^T \chi_\downarrow^*] \\
& + b^\dagger_{L\lambda}(\mathbf{p}) a^\dagger_{R\lambda'}(-\mathbf{p}) [i \delta_{\lambda\uparrow} \delta_{\lambda'\uparrow} \chi_\downarrow^T \chi_\downarrow^*] \\
& + b^\dagger_{L\lambda}(\mathbf{p}) b_{R\lambda'}(\mathbf{p}) [-\delta_{\lambda\uparrow} \delta_{\lambda'\downarrow} \chi_\downarrow^T \chi_\uparrow^*] \Big\} \\
= i \int_p \Big\{ & b^\dagger_{L\uparrow}(\mathbf{p}) a^\dagger_{R\uparrow}(-\mathbf{p}) - a_{L\downarrow}(\mathbf{p}) b_{R\downarrow}(-\mathbf{p}) \Big\},
\end{aligned} \tag{B.51}$$

in which the spinor identities $[\chi_\downarrow^T \chi_\uparrow^*] = 0, [\chi_\downarrow^T \chi_\downarrow^*] = 1$ were used in the last equality.

Note that $\Psi_L^\dagger \Psi_R = [\Psi_L^T \Psi_R^*]^*$, if we take the conjugation as the non-Grassmann - conjugation, which means that we can find all of the cross-terms from complex- and hermitian conjugating (B.51). The result is

$$\begin{aligned}
H_{m_D} &= -\frac{1}{2} m_D \int_x \Psi_L^\dagger \Psi_R - \Psi_L^T \Psi_R^* + h.c. \\
&= i m_D \int_p \Big\{ b^\dagger_{L\uparrow}(\mathbf{p}) a^\dagger_{R\uparrow}(-\mathbf{p}) - a_{L\downarrow}(\mathbf{p}) b_{R\downarrow}(-\mathbf{p}) \Big\} + h.c.
\end{aligned} \tag{B.52}$$

Diagonalizing the mass terms

For this case, the appropriate rotation is

$$\begin{aligned}
\begin{pmatrix} b_{L\uparrow}(\mathbf{p}) \\ a_{R\uparrow}(\mathbf{p}) \end{pmatrix} &= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} a_{1\uparrow}(\mathbf{p}) \\ a_{2\uparrow}(\mathbf{p}) \end{pmatrix}, \\
\begin{pmatrix} b_{R\downarrow}(\mathbf{p}) \\ a_{L\downarrow}(\mathbf{p}) \end{pmatrix} &= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} a_{1\downarrow}(\mathbf{p}) \\ a_{2\downarrow}(\mathbf{p}) \end{pmatrix},
\end{aligned} \tag{B.53}$$

where the rotation angle is defined via

$$\tan 2\theta = \frac{2m_D}{m_R}. \tag{B.54}$$

Applying this to the sum $H_{m_D} + H_{m_R}$ yields

$$\begin{aligned} H_{\text{mass}} = \frac{i}{2}\int_p \Big\{ &[m_R \cos^2(\theta) + 2m_D \cos(\theta)\sin(\theta)]\Big(a_{1\downarrow}(\mathbf{p})a_{1\downarrow}(-\mathbf{p}) + a_{2\uparrow}(\mathbf{p})a_{2\uparrow}(-\mathbf{p})\Big) \\ &+ [m_R \sin^2(\theta) - 2m_D \cos(\theta)\sin(\theta)]\Big(a_{1\uparrow}(\mathbf{p})a_{1\uparrow}(-\mathbf{p}) + a_{2\downarrow}(\mathbf{p})a_{2\downarrow}(-\mathbf{p})\Big)\Big\} + h.c., \end{aligned} \tag{B.55}$$

in which we identify masses

$$\begin{aligned} m_1 &= m_R \cos^2(\theta) + 2m_D \cos(\theta)\sin(\theta), \\ m_2 &= |m_R \sin^2(\theta) - 2m_D \cos(\theta)\sin(\theta)|, \end{aligned} \tag{B.56}$$

and relabel our operators as

$$\begin{aligned} c_{1\lambda} &= \begin{cases} a_{2\uparrow}, & \lambda =\uparrow \\ a_{1\downarrow}, & \lambda =\downarrow \end{cases} \\ c_{2\lambda} &= \begin{cases} a_{1\uparrow}, & \lambda =\uparrow \\ a_{2\downarrow}, & \lambda =\downarrow \end{cases} \end{aligned} \tag{B.57}$$

We remark that this rotation and relabeling keeps the form of the kinetic terms intact. For the mass terms, we have thus found

$$\begin{aligned} H_{\text{mass}} &= \frac{i}{2}\int_p \Big\{ m_1\Big(c_{1\uparrow}(\mathbf{p})c_{1\uparrow}(-\mathbf{p}) + c_{1\downarrow}(\mathbf{p})c_{1\downarrow}(-\mathbf{p})\Big) \\ &\qquad + m_2\Big(c_{2\uparrow}(\mathbf{p})c_{2\uparrow}(-\mathbf{p}) + c_{2\downarrow}(\mathbf{p})c_{2\downarrow}(-\mathbf{p})\Big)\Big\} + h.c. \\ &= \frac{i}{2}\sum_{i,\lambda} m_i \int_p c_{i\lambda}(\mathbf{p})c_{i\lambda}(-\mathbf{p}) + c^\dagger_{i\lambda}(\mathbf{p})c^\dagger_{i\lambda}(-\mathbf{p}), \end{aligned} \tag{B.58}$$

which is indeed the same shape as we found earlier in the two-flavor Majorana case, (B.27). From here on out, the analysis of the seesaw case is identical to what has been presented from eqns. (B.28) to (B.35).

Lastly, inverting the Bogoliubov transformation results in

$$\begin{aligned} b_{R\downarrow}(\mathbf{p}) &= \cos(\theta)\Big(\alpha_{1p}A_{1\downarrow}(\mathbf{p}) - \beta_{1p}A^\dagger_{1\downarrow}(-\mathbf{p})\Big) + \sin(\theta)\Big(\alpha_{2p}A_{2\downarrow}(\mathbf{p}) - \beta_{2p}A^\dagger_{2\downarrow}(-\mathbf{p})\Big), \\ a_{L\downarrow}(\mathbf{p}) &= \cos(\theta)\Big(\alpha_{2p}A_{2\downarrow}(\mathbf{p}) - \beta_{2p}A^\dagger_{2\downarrow}(-\mathbf{p})\Big) - \sin(\theta)\Big(\alpha_{1p}A_{1\downarrow}(\mathbf{p}) - \beta_{1p}A^\dagger_{1\downarrow}(-\mathbf{p})\Big), \end{aligned} \tag{B.59}$$

and similar for the operators with helicity up.

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